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A Theoretical Approach to Room Acoustic Simulations Based on a Radiative Transfer Model

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Summary

A theoretical approach to room acoustic simulations based on a radiative transfer model is developed by adapting the classical radiative transfer theory from optics to acoustics. The proposed acoustic radiative transfer model expands classical geometrical room acoustic modeling algorithms by incorporating a propagation medium that absorbs and scatters radiation, handling both diffuse and non-diffuse reflections on boundaries and objects in the room. The main scope of this model is to provide a proper foundation for a wide number of room acoustic simulation models, in order to establish and unify their principles. It is shown that this room acoustic modeling technique establishes the basis of two recently proposed algorithms, the acoustic diffusion equation and the room acoustic rendering equation. Both methods are derived in detail using an analytical approximation and a simplified integral equation of the proposed method, respectively, allowing a clear definition of the underlying assumptions, limitations, advantages and disadvantages.

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1. Introduction

In this paper a theory is developed for acoustic radiative transfer modeling that generalizes a wide variety of room acoustic modeling algorithms. The classical radiative transfer theory from optics [1] is adapted for acoustics. The proposed theory expands classical geometrical room acoustics modeling algorithms by incorporating a propagation medium that absorbs and scatters radiation, handling both diffuse and non-diffuse reflections both on boundaries and on objects within the room. From this theory, the concept of sound particles arises for modeling acoustic energy transfer by radiation phenomena occurring in a room. An integro-differential equation describes the interaction of the sound particles with the medium.

Equations of radiative transfer have application in a wide variety of areas, e.g., illuminating engineering [2], radiative heat transfer [3], remote sensing [4], vision [5], and computer graphics [6]. Since radiative transfer theory is based on conservation of energy [7], as the acoustic wave propagation theory is, its use in acoustics remains as a valid assumption. Hence, an energy balance model based on the radiative transfer of sound particles is proposed. As a general model, this room acoustic modeling technique establishes the basis of two recently proposed algorithms, the *acoustic diffusion equation* [8] and the *room acoustic rendering equation* [9], which also generalizes widely used geometrical room acoustics modeling methods, such as the image source [10], ray tracing [11], beam tracing [12], and acoustic radiosity [13]. Considerable attention has been given to these methods recently. In this paper, both methods are derived using an analytical approximation and a simplified integral equation, respectively. As a result, both methods are shown to be a particularization of the acoustic radiative transfer model.

The paper is organized as follows: first the radiative transfer theory is reviewed, its assumptions are described, and the acoustic radiative transfer model for room acoustic simulations is derived. Next, both the acoustic diffusion equation model, and the room acoustic rendering equation are reviewed and derived from the proposed general model. Finally, the underlying assumptions, limitations, advantages and disadvantages of each method are discussed.

2. Acoustic radiative transfer model

In this section a mathematical description of the transport of sound particles in a room is proposed by means of an analogy with the radiative transfer theory for transport of photon energy in a scattering medium. First, a general overview of the mathematical concepts of the radia-

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tive transfer theory is presented. Next, their relations with sound particle transport are described together with the definition of the acoustic radiative transfer equation. Finally, in order to obtain a complete model for rooms of complex shape, a general boundary condition definition is presented. The mathematics of this paper is based on previous work on light transport in turbid or scattering media [14, 15, 16].

2.1. General assumptions

Radiative transfer theory is the study of energy transfer in the form of electromagnetic radiation, i.e., light. The propagation of radiation through a medium is mainly affected by absorption, emission and scattering processes.

Sound, just as light, is a wave phenomenon. There are several differences between light and sound, including a much lower propagation speed of sound and correspondingly longer wavelengths, and the absence of polarization. However, in spite of these differences, similarities between light and sound phenomena have made it possible to develop well-established modeling techniques in room acoustics, known as *geometrical acoustics* [17]. In a scattering medium light is scattered and absorbed due to inhomogeneities and absorption characteristics of the medium, and in room acoustics the sound field is usually modeled by sound particles with the same constant energy, propagating along straight lines and striking walls or scattering objects which absorb and scatter as well [18, 19].

The radiative transfer theory requires some assumptions concerning the nature of the particles, thus, certain applicability conditions must be fulfilled:

- The particles are so small and numerous that their statistical distribution can be treated as a continuum.
- At any point in time a particle is completely characterized by its position and velocity, and internal states such as frequency for sound particles.
- The approximation of geometrical optics, where the wavelength is essentially shorter than the scale of variation of a macrosystem's parameters, uses the concepts of rays or beams for wave propagation in a medium.
- The relationship between the size of individual particles and the working wavelength is arbitrary.
- All processes of the interaction of the external field with a unit volume of a medium are reduced to three acts only - absorption, emission and scattering.
- Particles do not interact with one another; or more precisely, such interactions are negligible, i.e. it is necessary to ignore effects such as interference between sound particles.

2.2. Basic variables of the theory

In this section, an overview of the variables involving the radiative transfer equation is presented and the usual notation recognized in radiative optics field is introduced [14].

The assumptions above requires the definition of *phase space* as an abstraction for dealing with configurations of particles. For simplicity, particles that move with isotropic

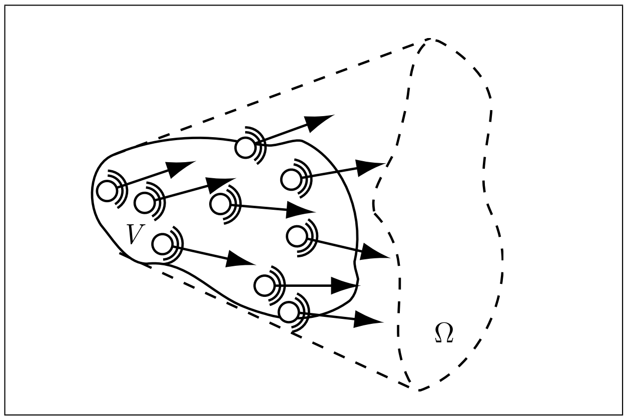

Figure 1. A five-dimensional volume in phase space as a result of the cartesian product of a volume V and a solid angle Ω.

constant speed c and without internal states are considered. This restricted problem corresponds to assuming homogeneous propagation and reflection behavior of sound radiation within the frequency band under analysis. Five degrees of freedom per particle are normally used, corresponding to a 5-dimensional phase space $R^3 \times S^2$, where R^3 is the Euclidean 3-space and S^2 is the unit sphere in R^3. To derive a balance equation for particles, the environment and the basic variables of the particles within some fixed volume in phase space must be defined. The room is defined by its total volume V, and its total surface S_t. Let $V \subset R^3$ be an arbitrary volume, and let $\Omega \subset S^2$ be an arbitrary solid angle, as showed in Figure 1. Let us consider the cartesian product $V \times \Omega$, which defines a set of all pairs $(\mathbf{r}, \hat{\mathbf{s}})$ such that $\mathbf{r} \in V$ and $\hat{\mathbf{s}} \in \Omega$. This is directly related with the transport of power along a direction $\hat{\mathbf{s}}$ (where $\hat{}$ denotes a unit vector) through an absorbing, emitting and scattering infinitesimal volume element dV around the point $\mathbf{r}$.

To describe a set of particles completely we must specify all degrees of freedom of each particle at an instant in time; this corresponds to discrete points in phase space. There are many sound particles moving in the enclosure at the same time. For a continuum of particles their distribution in the phase space and the evolution of the time distribution is better described in terms of the sound particle *phase space density*. It is denoted by $N(\mathbf{r}, \hat{\mathbf{s}}, t)$ and defined over phase space and time as the number of particles in a infinitesimal volume dV about the point $\mathbf{r}$ moving in a direction within a infinitesimal solid angle $d\Omega$ about $\hat{\mathbf{s}}$ at time t. To cover the sound particle transport problem with the phase space density, the sound particle energy attribute must be incorporated in it, so that the units are $\mathrm{Jm^{-3}sr^{-1}}$. Therefore, the total energy in this phase space volume is obtained by

$$N(t) = \int_\Omega \int_V N(\mathbf{r}, \hat{\mathbf{s}}, t) \, \mathrm{d}V \, \mathrm{d}\Omega. \tag{1}$$

Our task will be to determine how the number of particles in $V \times \Omega$ changes with time. The *sound energy density* is then obtained, and integrating this density function over solid angle Ω, gives the average number of sound particles per unit volume (Jm^{-3}),

$$w(\mathbf{r},t) = \int_{\Omega} N(\mathbf{r},\hat{\mathbf{s}},t)\,\mathrm{d}\Omega. \quad (2)$$

Moreover, the current density or *sound energy flow vector* $\mathbf{J}$ (Wm^{-2}) is defined as the net energy flow per unit area per unit time in a certain direction $\hat{\mathbf{s}}$,

$$\mathbf{J}(\mathbf{r},t) = \int_{\Omega} \hat{\mathbf{s}} c N(\mathbf{r},\hat{\mathbf{s}},t)\,\mathrm{d}\Omega, \quad (3)$$

which is the vector counterpart of fluence rate pointing in the direction of the prevalent energy flow.

But rather than focusing on the particles within a volume, the transfer theory emphasizes the energy that propagates through or is emitted from a particular surface dA and the rate at which particles cross it by defining *sound radiance* $L(\mathbf{r},\hat{\mathbf{s}},t)$. The relation between phase space density and radiance is

$$L(\mathbf{r},\hat{\mathbf{s}},t) = cN(\mathbf{r},\hat{\mathbf{s}},t), \quad (4)$$

its units are $\mathrm{Wm}^{-2}\mathrm{sr}^{-1}$, and it will be the dependent variable in the acoustic radiative transfer equation. It is defined as energy flow per unit normal area per unit solid angle per unit time, where the normal area is perpendicular to the flow direction. In some cases, it is an useful abstraction to allow surfaces to be emitters. Sound that is emitted from boundaries can be described in terms of *sound emittance* as an area source by $L_0(\mathbf{r}_b,\hat{\mathbf{s}},t)$ where $\mathbf{r}_b$ is a surface point.

To take into account the creation of new particles by sources per unit volume per unit solid angle and per unit time a *phase space source term* is expressed by $q(\mathbf{r},\hat{\mathbf{s}},t)$, and it carries the unit of $\mathrm{Wm}^{-3}\mathrm{sr}^{-1}$.

In their travel through the medium the sound particles encounter scattering objects. The collisions between particles are neglected in the radiative transfer equation assumptions (see section 2.1 for details), and only the collisions between particles and scattering objects exist. Scattering objects give rise to both absorption and reflection, like walls and furniture in a room, or air absorption. In radiative transfer theory properties of the medium are defined as an attenuation proportional to the path length by absorption, scattering and emission.

For simplicity, let us assume an average value in the definition of acoustic properties of the medium. The average straight-line distance experienced by a sound particle between collisions in a highly scattering medium is the sound particle *mean free path* λ. Moreover, when a collision with a scattering object occurs, the particle is reflected with the energy $(1-\bar{\alpha})$ and is absorbed with the energy $\bar{\alpha}$, where $\bar{\alpha}$ is defined in acoustics as the room mean sound energy absorption coefficient. Note that it not only refers to absorption of surfaces but also to absorption of objects in the enclosure. Therefore, in our analogy the acoustical properties of the medium are defined as follows. The fraction of incident radiation scattered over the unit of the mean free path is designated by μ_s,

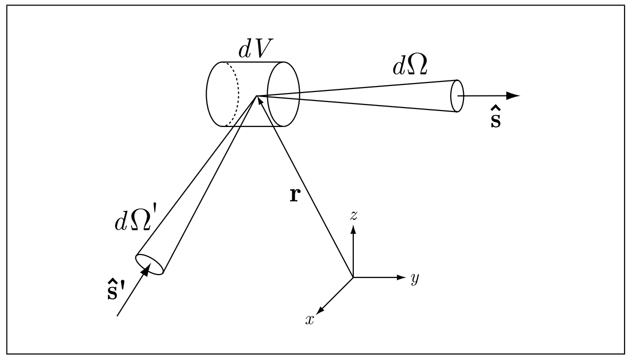

Figure 2. Schematic of a stationary infinitesimal volume element.

$$\mu_s = \frac{(1-\bar{\alpha})}{\lambda}. \quad (5)$$

To take account of the absorption of the scattering objects and the medium, the fraction of incident radiation absorbed over the unit of the mean free path is designated by μ_a,

$$\mu_a = m + \frac{\bar{\alpha}}{\lambda}, \quad (6)$$

where m is the air absorption attenuation [20] and all values have the dimension of m^{-1}.

2.3. The acoustic radiative transfer equation

In this section we approach an acoustic radiative transfer equation focusing on sound particles propagating at a finite speed. In this case, the time dependence cannot be neglected, as is usually done in light energy transport theory. We assume linear acoustics, and the environment for the sound propagation is described by using the concepts established in section 2.1 and section 2.2. The sound field carries energy, and propagating sound energy may be compared to propagating electromagnetic radiation by denoting it *sound radiance*.

Using the quantities defined above, the acoustic radiative transfer equation can heuristically be derived from the principle of conservation of energy. A beam of sound can be defined as a set of traveling sound particles moving in a direction within a solid angle $d\Omega$ around a direction $\hat{\mathbf{s}}$ and is usually represented as a pencil as shown in Figure 2. Let us consider a stationary infinitesimal volume element dV and all possible contributions to the energy change in this volume element within a infinitesimal solid angle element $d\Omega$ around the outward direction $\hat{\mathbf{s}}$. In addition, $d\Omega'$ is a infinitesimal solid angle element around an incoming direction $\hat{\mathbf{s}}'$. The radiative transfer equation states that a beam of sound loses energy through divergence, dN_{div}, and attenuation, including both absorption, dN_{abs}, and scattering, $dN_{\mathrm{l_{sca}}}$, away from the beam, and gains energy from scattering, $dN_{\mathrm{g_{sca}}}$, directed towards the beam and sound sources, dN_{src}, in the medium.

The time-resolved transfer equation is a mathematical expression of the build-up of the sound particle density function. Therefore, the changes in energy in the volume element per unit time, dN, is expressed by

$$\mathrm{d}N = \int_\Omega \int_V \frac{1}{c}\frac{\partial L(\mathbf{r},\hat{\mathbf{s}},t)}{\partial t}\,\mathrm{d}V\,\mathrm{d}\Omega. \tag{7}$$

This rate of change is a result of the balance between the three negative loss contributions and the two positive gain contributions mentioned above. The principle of conservation of energy requires that

$$\mathrm{d}N = -\mathrm{d}N_{\mathrm{div}} - \mathrm{d}N_{\mathrm{l_{sca}}} - \mathrm{d}N_{\mathrm{abs}} + \mathrm{d}N_{\mathrm{g_{sca}}} + \mathrm{d}N_{\mathrm{src}}. \tag{8}$$

The first right-hand term of equation (8), dN_{div}, takes into account the sound particles with directions in Ω that either escape from or enter into the volume V simply by streaming. In general, some infinitesimal surface patches will have positive flows and some negative.

More precisely, the change due to streaming is the net flow of particles with directions in Ω that pass through the surface A of volume V, as shown in Figure 3. The streaming through each infinitesimal patch on the surface dA depends only on the flux that is normal $\hat{\mathbf{n}}$ to the patch. The change in energy due to streaming through A can be expressed by integrating the normal component of the sound radiation due to particles in Ω over the entire surface A,

$$\mathrm{d}N_{\mathrm{div}} = \int_\Omega \int_A L(\mathbf{r},\hat{\mathbf{s}},t)(\hat{\mathbf{s}}\cdot\hat{\mathbf{n}})\,\mathrm{d}A\,\mathrm{d}\Omega. \tag{9}$$

Using Gauss's theorem we can convert the single surface integral into a volume integral, yielding

$$\mathrm{d}N_{\mathrm{div}} = \int_\Omega \int_V \hat{\mathbf{s}}\cdot\nabla L(\mathbf{r},\hat{\mathbf{s}},t)\,\mathrm{d}V\,\mathrm{d}\Omega. \tag{10}$$

Here, in addition to Gauss's theorem, we have used the fact that

$$\nabla\cdot\left[\hat{\mathbf{s}}L(\mathbf{r},\hat{\mathbf{s}},t)\right] = \hat{\mathbf{s}}\cdot\nabla L(\mathbf{r},\hat{\mathbf{s}},t). \tag{11}$$

This contribution is due to local beam propagation without collisions with scattering objects; thus, it can exist even in a non-scattering medium.

The next two right-hand terms of equation (8), $dN_{\mathrm{l_{sca}}}$ and dN_{abs}, represent the energy change per unit time in the volume element within the solid angle element due to collisions, which can be separated into scattering and absorption events. Thus, the term $dN_{\mathrm{l_{sca}}}$ takes account of the sound particles scattered from direction $\hat{\mathbf{s}}$ to any another direction,

$$\mathrm{d}N_{\mathrm{l_{sca}}} = \int_\Omega \int_V \mu_s L(\mathbf{r},\hat{\mathbf{s}},t)\,\mathrm{d}V\,\mathrm{d}\Omega, \tag{12}$$

where μ_s is the scattering coefficient defined in equation (5).

The absorbed sound particles coming from direction $\hat{\mathbf{s}}$ are expressed in the next term, dN_{abs},

$$\mathrm{d}N_{\mathrm{abs}} = \int_\Omega \int_V \mu_a L(\mathbf{r},\hat{\mathbf{s}},t)\,\mathrm{d}V\,\mathrm{d}\Omega, \tag{13}$$

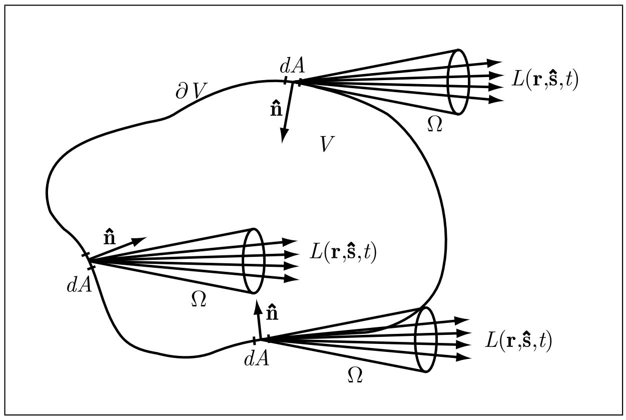

Figure 3. Integrating the flux due to Ω over the surface ∂V.

where μ_a is the absorption coefficient in equation (6); both air absorption and absorption by scattering objects are included.

Sound particles gained through scattering from any direction $\hat{\mathbf{s}}'$ into $d\Omega$ around direction $\hat{\mathbf{s}}$ per unit time constitute the fourth term $dN_{\mathrm{g_{sca}}}$,

$$\mathrm{d}N_{\mathrm{g_{sca}}} = \int_\Omega \int_V \mu_s \int_{\Omega'} P(\hat{\mathbf{s}}',\hat{\mathbf{s}})L(\mathbf{r},\hat{\mathbf{s}},t)\,\mathrm{d}\Omega'\,\mathrm{d}V\,\mathrm{d}\Omega, \tag{14}$$

where $P(\hat{\mathbf{s}}',\hat{\mathbf{s}})$ is the phase function that represents the probability of particles with propagation direction $\hat{\mathbf{s}}'$ being scattered into the solid angle $d\Omega$ around $\hat{\mathbf{s}}$. The simplest phase function is the isotropic phase function where $P(\hat{\mathbf{s}}',\hat{\mathbf{s}}) = 1/4\pi$.

Finally, energy produced by a source in the volume element within the solid angle element per unit time is given by the last term dN_{src},

$$\mathrm{d}N_{\mathrm{src}} = \int_\Omega \int_V q(\mathbf{r},\hat{\mathbf{s}},t)\,\mathrm{d}V\,\mathrm{d}\Omega, \tag{15}$$

where $q(\mathbf{r},\hat{\mathbf{s}},t)$ is an omnidirectional volume source term.

All six terms derived in equation (8) entail integration over both V and Ω. The equality must hold for both integrands because $V\times\Omega$ is arbitrary, so the two outer integrals can be removed. Finally, the radiative transfer equation is derived,

$$\begin{aligned}\frac{1}{c}\frac{\partial L(\mathbf{r},\hat{\mathbf{s}},t)}{\partial t} &= -\hat{\mathbf{s}}\cdot\nabla L(\mathbf{r},\hat{\mathbf{s}},t) - \mu_t L(\mathbf{r},\hat{\mathbf{s}},t) \\ &\quad + \mu_s \int_{\Omega'} P(\hat{\mathbf{s}}',\hat{\mathbf{s}})L(\mathbf{r},\hat{\mathbf{s}}',t)\,\mathrm{d}\Omega' + q(\mathbf{r},\hat{\mathbf{s}},t),\end{aligned} \tag{16}$$

where μ_t is the attenuation coefficient as the sum of the absorption coefficient μ_a and the scattering coefficient μ_s. The last equation is the *acoustic radiative transfer equation*. For time-independent responses, valid for steady-state situation, the left-hand side of equation (16) is zero, which is normally used in computer graphics due to the higher speed of light. It should be noted that this equation is similar to the one obtained in [21]. However this paper presents a more comprehensive and detailed exposition of the acoustic radiative transfer equation.

2.4. Boundary conditions of the radiative transfer equation

The presence of the gradient operator in equation (16), makes it a first-order differential equation in the spatial variables for a fixed direction $\hat{\mathbf{s}}$. As such, the equation requires knowledge of the sound radiance at a single point in space into the direction $\hat{\mathbf{s}}$ to be a complete description of the physical situation.

Generally, the point where the sound radiance can be specified lies on the boundary of an enclosure surrounding the medium. Therefore, equation (16) is only valid away from the boundaries. The sound radiance leaving the surfaces must be determined by solving the boundary conditions.

To express the boundary conditions, let us denote ∂V the entire boundary surface that limits the entire medium V, and $\hat{\mathbf{n}}$ the surface normal at point $\mathbf{r}_b \in \partial V$. The S^2 space is partitioned into two hemispheres at each boundary point, designating the hemisphere where $(\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}) > 0$ the *positive hemisphere* by Ω^+ and analogously the *negative hemisphere* by Ω^- (see Figure 4 for details).

A complete definition of the boundary condition involves *explicit* and *implicit* boundary conditions. Explicit boundary conditions are independent of sound radiance and take account of the particles that are generated by independent processes $L_0(\mathbf{r}_b, \hat{\mathbf{s}}, t)$.

Implicit or reflecting boundary conditions introduce a physical interpretation of the scattering at the boundary: particles created at the boundary are the result of reflections of particles impacting on it. A common type of implicit boundary condition is given by

$$L(\mathbf{r}_b, \hat{\mathbf{s}}, t) = \int_{\Omega^-} R_F(\mathbf{r}_b; \hat{\mathbf{s}}', \hat{\mathbf{s}}) L(\mathbf{r}_b, \hat{\mathbf{s}}', t)(\hat{\mathbf{s}}' \cdot -\hat{\mathbf{n}})\, \mathrm{d}\Omega', \quad (17)$$

where R_F is the surface scattering or reflecting function with units of sr^{-1} defined as the probability that a particle at $\mathbf{r}_b$ moving in the $\hat{\mathbf{s}}'$ direction will be reflected into new direction $\hat{\mathbf{s}}$.

Now, we impose a general boundary condition, including both explicit and implicit terms,

$$\begin{aligned} L(\mathbf{r}_b, \hat{\mathbf{s}}, t) = {} & L_0(\mathbf{r}_b, \hat{\mathbf{s}}, t) \\ & + \int_{\Omega^-} R_F(\mathbf{r}_b; \hat{\mathbf{s}}', \hat{\mathbf{s}}) L(\mathbf{r}_b, \hat{\mathbf{s}}', t)(\hat{\mathbf{s}}' \cdot -\hat{\mathbf{n}})\, \mathrm{d}\Omega'. \end{aligned} \quad (18)$$

Finally, equation (18) together with the acoustic radiative transfer equation (16) compose the system of equations for sound particles transport in the acoustic radiative transfer model.

3. Derivation of the diffusion equation model

In this section the acoustic diffusion equation model is derived from an approximation to the acoustic radiative transfer model.

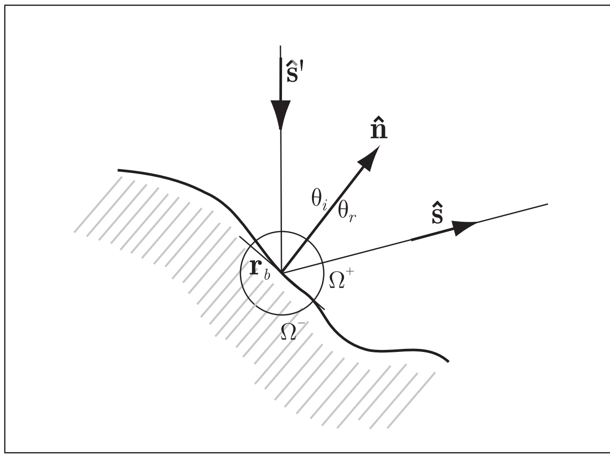

Figure 4. The surface ∂V partitions with the set of directions S^2 at each point $\mathbf{r}_b \in \partial V$.

3.1. Acoustic diffusion equation model

Valeau et al. [22] have recently proposed an alternative approach to predict the sound field in arbitrary rooms with a satisfactory accuracy and with low computation time based on a diffusion model [23, 8]. The acoustic diffusion equation method has been applied to model a wide variety of room types, such as single-space enclosure [22], long rooms [24], fitted rooms [25], and coupled rooms [26, 27, 28].

This model has also been used for sound field predictions in open spaces like city streets [29], and street canyons [30].

The model is based on the assumption that the sound propagation in rooms with diffusely reflecting boundaries can be modeled by analogy with propagation of gas particles in a diffusing fluid. A partial differential equation for this phenomenon is available in the technical literature. The scattering objects take the place of the walls and reflect the sound diffusely in the volume. In this case, the sound energy flow vector $\mathbf{J}(\mathbf{r}, t)$ within a room, is related to the gradient of the sound energy density $w(\mathbf{r}, t)$, according to Fick's law [8, 31],

$$\mathbf{J}(\mathbf{r}, t) = -D\nabla w(\mathbf{r}, t), \quad (19)$$

where D is the *diffusion coefficient*, which is expressed as

$$D = \frac{\lambda c}{3}. \quad (20)$$

This is a term that takes account of the room geometry and shape through the mean free path. In classical acoustic theory [32, 33], the mean free path for empty room is given by

$$\lambda = \frac{4V}{S_t}, \quad (21)$$

for volume V and interior surface area S_t of the enclosure under investigation. For a medium containing scattering objects the mean free path for the propagation of particles should be evaluated as in [25].

With an omnidirectional sound source $q_0(\mathbf{r}, t)$, in the volume the sound energy density $w(\mathbf{r}, t)$ distribution can be expressed as [22],

$$\begin{aligned}\frac{\partial w(\mathbf{r}, t)}{\partial t} &= -\nabla \cdot \mathbf{J}(\mathbf{r}, t) \\ &= D\nabla^2 w(\mathbf{r}, t) - \sigma w(\mathbf{r}, t) - mcw(\mathbf{r}, t) + q_0(\mathbf{r}, t) \text{ in } V,\end{aligned} \tag{22}$$

where $\sigma w(\mathbf{r}, t)$ describes the energy loss per unit volume due to absorption at the room boundaries, with $\sigma = \bar{\alpha}c/\lambda$ and $mcw(\mathbf{r}, t)$, takes account of the air absorption in the enclosure [34].

If the diffusion equation should be used in a room of arbitrary shape, the equivalent scattering medium has to be bounded with appropriate boundary conditions. In previous work two kinds of boundary conditions have been presented, a *homogeneous Neumann* boundary condition [22, 31] where sound energy cannot escape from the room boundaries (its validity is discussed in [35]); and a more realistic *mixed* boundary condition for general situations which allows energy exchanges with the boundaries ∂V.

In order to use the mixed boundary conditions, the absorption should occur on room surfaces rather than in the volume, which gives rise to the following system of equations [22, 34],

$$\frac{\partial w(\mathbf{r}, t)}{\partial t} - D\nabla^2 w(\mathbf{r}, t) + mcw(\mathbf{r}, t) = q_0(\mathbf{r}, t) \text{ in } V, \tag{23}$$

$$D\frac{\partial w(\mathbf{r}, t)}{\partial \hat{\mathbf{n}}} + cA_x w(\mathbf{r}, t) = 0 \quad \text{on } \partial V, \tag{24}$$

which is a second order parabolic partial differential equation with mixed boundary conditions, with $\hat{\mathbf{n}}$ denoting the surface outgoing normal. The absorption factor $A_x = A_x(\mathbf{r}, \alpha)$ in equation (24) (where α is the surface absorption coefficient) can describe rooms with low [22] or high absorption [36, 37].

Although specular reflection cannot be applied an empirical extension to this model for non-diffuse reflection has been developed [38].

3.2. Relation to acoustic radiative transfer equation

Despite the developments and the potential of the acoustic diffusion equation model, there is a need for a clear, complete exposition of the theory and the assumptions behind the method. The acoustic radiative transfer equation is difficult to solve since it has six independent variables ($\mathbf{r} = [x, y, z] \in R^3, \hat{\mathbf{s}} = [\theta, \phi] \in S^2, t$). By making appropriate assumptions about the behavior of sound particles in a scattering medium, the number of independent variables can be reduced. Many different approximations have been developed in several scientific fields trying to simplify the radiative transfer equation [39], and here it is adapted for room acoustic modeling. One of the most useful is the diffusion approximation since it can be solved analytically for simple geometries, and numerically for complex enclosures [15].

Two main assumptions are needed for the application of the diffusion theory to the radiative transfer equation:

1. The scattering density must be high, and the reflection of energy must dominate over absorption. This means that after numerous diffuse reflections the radiance becomes nearly isotropic. This assumption is sometimes called *directional broadening*.
2. In a primarily scattering medium the time for substantial energy flux changes is much longer than the time it takes to traverse one mean free path. Thus, over one transport mean free path, the fractional change in the sound energy flow vector is much less than unity [31]. This property is sometimes called *temporal broadening*.

As pointed out in section 3.1, other derivations of the acoustic diffusion equation model have been presented in prior literature where directional and temporal broadening have also been assumed. In the diffusion equation proposed by Picaut et al. [8] for room-acoustics, a first order approximation in particle velocity of the particle density function is used. For the expansion to be valid the scattering obstacles are supposed to be in more quantity than the sound particles, which is in accordance with the directional broadening. Moreover, the variation in the energy density and the energy flow per mean free path must be small, which is the temporal broadening assumption. Otherwise, in an alternative approximation of the diffusion equation based on the transport equation [30], which is used for predicting sound propagation in a street canyon, the two planes that compose the street are closer to each other, increasing the particle frequency collision on the planes. It should be noted that this assumption is the same as the directional broadening. Moreover, the time variable is rescaled, which is related to the temporal broadening

A formal method of solving the resulting integro-differential equation is to expand the sound radiation in appropriate function series. In order to obtain the acoustic diffusion equation, this is done by spherical harmonics $Y_{n,m}$ [40]. Thus the position and directional variables can be separated into new functions.

Expanding the sound radiance function $L(\mathbf{r}, \hat{\mathbf{s}}, t)$ yields:

$$L(\mathbf{r}, \hat{\mathbf{s}}, t) \approx \sum_{n=0}^{1} \sum_{m=-n}^{+n} L_{n,m}(\mathbf{r}, t) Y_{n,m}(\hat{\mathbf{s}}), \tag{25}$$

where $L_{n,m}$ are the expansion coefficients. In the diffusion approximation only the lowest order in the expansion are used, to meet the directional broadening assumption. The term for $n = 0$ and $m = 0$ represents the isotropic component, whereas the terms for $n = 1$ and $m = 0, \pm 1$ represent the anisotropic component.

Substituting equation (25) into equation (2) yields

$$w(\mathbf{r}, t) = \frac{4}{c}\pi L_{0,0}(\mathbf{r}, t)\, Y_{0,0}(\hat{\mathbf{s}}). \tag{26}$$

Multiplying equation (25) by $\hat{\mathbf{s}}$ and substituting it into equation (3) gives

$$\mathbf{J}(\mathbf{r}, t) \cdot \hat{\mathbf{s}} = \frac{4\pi}{3} \sum_{m=-1}^{1} L_{1,m}(\mathbf{r}, t)\, Y_{1,m}(\hat{\mathbf{s}}), \tag{27}$$

Combining equations (26) and (27) with the expansion of $L(\mathbf{r}, \hat{\mathbf{s}}, t)$ equation (25) gives

$$L(\mathbf{r}, \hat{\mathbf{s}}, t) = \frac{c}{4\pi} w(\mathbf{r}, t) + \frac{3}{4\pi} \mathbf{J}(\mathbf{r}, t) \cdot \hat{\mathbf{s}}. \quad (28)$$

The source function can also be expanded in spherical harmonics,

$$q(\mathbf{r}, \hat{\mathbf{s}}, t) = \frac{1}{4\pi} q_0(\mathbf{r}, t) + \frac{3}{4\pi} \mathbf{q}_1(\mathbf{r}, t) \cdot \hat{\mathbf{s}}. \quad (29)$$

Now, substituting this diffusion approximations, equation (28) and equation (29), into the acoustic radiative transfer equation, equation (16), and integrating over the full solid angle gives the following scalar differential equation,

$$\frac{1}{c}\frac{\partial w(\mathbf{r}, t)}{\partial t} + \frac{1}{c}\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \mu_a w(\mathbf{r}, t) = \frac{1}{c} q_0(\mathbf{r}, t). \quad (30)$$

Finally, substituting equations (28) and (29) into equation (16), multiplying both sides by $\hat{\mathbf{s}}$ and integrating over all solid angle yields the vector differential equation

$$\frac{1}{c^2}\frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t} + \frac{1}{3}\nabla w(\mathbf{r}, t) + \frac{1}{3D}\mathbf{J}(\mathbf{r}, t) = \frac{1}{c}\mathbf{q}_1(\mathbf{r}, t). \quad (31)$$

where D is the diffusion coefficient defined as

$$D = \frac{c}{3\mu_t} = \frac{c}{3(\frac{1}{\lambda} + m)} \approx \frac{\lambda c}{3}. \quad (32)$$

Note that this formulation of the diffusion coefficient is in accordance with the extension of the diffusion model presented in [34], and the approximation in the equation corresponds to neglecting the air absorption, in accordance with [22] (see equation (20)).

It should be emphasized that in this approach the phase function in equation (16) is regarded as isotropic,

$$\int_{4\pi} P(\hat{\mathbf{s}}', \hat{\mathbf{s}})\, \mathrm{d}\Omega' = 1; \quad (33)$$

therefore, all boundaries are assumed to be diffusely reflecting in order to fulfill directional broadening.

The source term $q(\mathbf{r}, \hat{\mathbf{s}}, t)$ is also supposed to be isotropic, i.e. omnidirectional, therefore $\mathbf{q}_1(\mathbf{r}, \hat{\mathbf{s}}, t) = 0$. Finally, assuming that the fractional change in energy flow within a mean free time (reverse of mean free path) is small, being this supposition what is previously called temporal broadening, equation (31) in the steady state yields an expression for the sound energy flow vector that is analogous to Fick's law (see equation (19)). Even when the source varies in time, equation (19) is a good approximation if temporal broadening can be assumed [31].

The two relations between the sound energy density and the sound energy flow vector (see equations (30) and (19)) can be combined, which leads to the *diffusion equation*,

$$\frac{\partial w(\mathbf{r}, t)}{\partial t} - D\nabla^2 w(\mathbf{r}, t) + c\mu_a w(\mathbf{r}, t) = q_0(\mathbf{r}, t), \quad (34)$$

where the diffusion coefficient is supposed to be space-invariant. Note that this expression is in accordance with equation (22) using the definition of μ_a (see equation 6).

In this derivation of the diffusion equation from the acoustic radiative transfer equation, two approximations have been made,

- The directional broadening permits the expansion of the radiance to be limited to the first-order spherical harmonics.
- The temporal broadening ensures that the fractional change in $\mathbf{J}(\mathbf{r}, t)$ in one transport mean free path is much less than unity.

The diffusion equation model can be regarded as an acoustic geometrical method, in the same way as ray-tracing and image-source models.

3.3. Boundary conditions of the diffusion equation

The diffusion equation derived in section 3.2 is valid for an infinite scattering medium. Consideration of boundary conditions permits the use of the diffusion equation to characterize sound propagation in a bounded medium.

In what follows two kinds of boundary conditions are considered, the homogeneous Neumann boundary condition and the mixed boundary condition but from the point of view of the diffusion approximation presented above. Figure 4 illustrates the coordinates in this section for sound reflection and absorption on a boundary.

3.3.1. Homogeneous boundary conditions

The absorption of the surfaces of the room are included in the diffusion equation expressed on equation (34) by the term μ_a. This term contains the loss of energy in the room per unit volume and per unit time associated with the mean room absorption coefficient $\bar{\alpha}$. Therefore the boundary condition should impose no flux of sound energy through the room surfaces. This boundary condition is expressed by taking into account that the direction-integrated sound radiation function at the boundary and directed towards the medium is equal to the direction-integrated sound radiation function at the boundary and directed out of the medium. In other words, total reflection yields

$$\int_{\Omega^+} L(\mathbf{r}_b, \hat{\mathbf{s}}, t)(\hat{\mathbf{s}} \cdot \hat{\mathbf{n}})\, \mathrm{d}\Omega = \int_{\Omega^-} L(\mathbf{r}_b, \hat{\mathbf{s}}', t)(\hat{\mathbf{s}}' \cdot -\hat{\mathbf{n}})\, \mathrm{d}\Omega'. \quad (35)$$

The diffusion approximation gives an expression for the sound radiation function equation (28), and the boundary condition becomes after integration

$$\frac{w(\mathbf{r}_b, t)}{4} + \frac{1}{2c}\mathbf{J}(\mathbf{r}_b, t) \cdot \hat{\mathbf{n}} = \frac{w(\mathbf{r}_b, t)}{4} - \frac{1}{2c}\mathbf{J}(\mathbf{r}_b, t) \cdot \hat{\mathbf{n}}. \quad (36)$$

Substituting Fick's law, equation (19), gives

$$\mathbf{J}(\mathbf{r}_b, t) \cdot \hat{\mathbf{n}} = -D\nabla w(\mathbf{r}_b, t) \cdot \hat{\mathbf{n}} = 0, \quad (37)$$

therefore,

$$\frac{\partial w(\mathbf{r}_b, t)}{\partial \hat{\mathbf{n}}} = 0 \quad \text{on} \quad \partial V. \quad (38)$$

This derivation yields equation (38) for boundaries as the homogeneous model described in Valeau et al. [22].

3.3.2. Mixed boundary conditions

The boundary condition introduced in section 3.3.1, has no real physical meanings for the application to complex rooms, because it assumes that the absorption occurs in the volume rather than on the room surfaces. Also, a boundary condition that makes it possible to define different absorption coefficients for each surface that forms the room is needed for simulating realistic enclosures. To model the local effects on the sound field induced by different absorbing materials on surfaces, let us adopt the following mixed boundary conditions based on partial-current boundary condition in light diffusion [16].

This boundary condition is derived introducing the surface reflectance factor R_F in equation (35),

$$\int_{\Omega^+} L(\mathbf{r}_b, \hat{\mathbf{s}}, t)(\hat{\mathbf{s}} \cdot \hat{\mathbf{n}})\, \mathrm{d}\Omega = \int_{\Omega^-} R_F(\mathbf{r}_b; \hat{\mathbf{s}}', \hat{\mathbf{s}}) L(\mathbf{r}_b, \hat{\mathbf{s}}', t)(\hat{\mathbf{s}}' \cdot -\hat{\mathbf{n}})\, \mathrm{d}\Omega'. \tag{39}$$

Evaluating the integrals after substitution with the diffusion approximation gives

$$\frac{w(\mathbf{r}_b, t)}{4} + \frac{1}{2c}\mathbf{J}(\mathbf{r}_b, t) \cdot \hat{\mathbf{n}} = R_F(\mathbf{r}_b)\frac{w(\mathbf{r}_b, t)}{4} - R_F(\mathbf{r}_b)\frac{1}{2c}\mathbf{J}(\mathbf{r}_b, t) \cdot \hat{\mathbf{n}}, \tag{40}$$

where $R_F(\mathbf{r}_b)$ is assumed to be independent of the angle of incidence following the diffusion approximation assumptions. Applying Fick's law gives

$$cw(\mathbf{r}_b, t) = -2D\frac{1 + R_F(\mathbf{r}_b)}{1 - R_F(\mathbf{r}_b)}\frac{\partial w(\mathbf{r}_b, t)}{\partial \hat{\mathbf{n}}}. \tag{41}$$

By analogy with room acoustics, the reflectance factor or energy based reflectivity can be expressed using the absorption coefficient as $(1 - \alpha(\mathbf{r}_b))$. Therefore, a simple operation after substitution yields

$$D\frac{\partial w(\mathbf{r}_b, t)}{\partial \hat{\mathbf{n}}} + \frac{c\alpha(\mathbf{r}_b)}{2(2 - \alpha(\mathbf{r}_b))}w(\mathbf{r}_b, t) = 0 \quad \text{on} \quad \partial V. \tag{42}$$

This equation is called the *modified boundary conditions* and along with equation (34) without the absorption term μ_a in the volume it leads to a system of equations for the acoustic diffusion model within the volume and the boundaries (see equation (23) and equation (24)). Note that equation (42) agrees with the one previously derived for acoustic purpose by Jing and Xiang [35].

4. Derivation of the room acoustic rendering equation

In this section it is shown that the room acoustic rendering equation is a particular solution of the acoustic radiative transfer equation. There are general solutions of the radiative transfer equation in integral form, mainly when the quasi-steady state is assumed, something that is usually done in electromagnetic works. However in this section, a simplified time-dependent analytical solution is obtained in integral form with boundary conditions included.

4.1. Room acoustic rendering equation

Recently, a general geometrical room acoustic modeling method, called the room acoustic rendering equation, has been presented [9]. The formulation used by Siltanen et al. is expressed with the notation and variables defined in section 2 for clarity. This integral equation based method can handle both diffuse and non-diffuse reflections.

The integral equation is describing the time-dependent sound radiance $L(\mathbf{r}_b, \hat{\mathbf{s}}, t)$ leaving a point on a surface $\mathbf{r}_b$ in direction $\hat{\mathbf{s}}$ as a combination of the emitted time-dependent radiance $L_0(\mathbf{r}_b, \hat{\mathbf{s}}, t)$ and the amount of the reflected time-dependent radiance $L(\mathbf{r}'_b, \hat{\mathbf{s}}', t - \frac{(\mathbf{r}_b - \mathbf{r}'_b)\cdot\hat{\mathbf{s}}'}{c})$ arriving from all visible surfaces,

$$L(\mathbf{r}_b, \hat{\mathbf{s}}, t) = L_0(\mathbf{r}_b, \hat{\mathbf{s}}, t) + \int_{\partial V} R(\mathbf{r}_b; \hat{\mathbf{s}}', \hat{\mathbf{s}}) L\left(\mathbf{r}'_b, \hat{\mathbf{s}}', t - \frac{(\mathbf{r}_b - \mathbf{r}'_b) \cdot \hat{\mathbf{s}}'}{c}\right) \mathrm{d}A', \tag{43}$$

where term $R(\mathbf{r}_b; \hat{\mathbf{s}}', \hat{\mathbf{s}})$ is the so-called *reflection kernel* that combines the reflectance factor $R_F(\mathbf{r}_b; \hat{\mathbf{s}}', \hat{\mathbf{s}})$, [1] the propagation operator, the geometry term, and the visibility term.

The formulation of equation (43) was adopted from computer graphics [41] as a time-dependent extension, using a direct analogy of parameters between optics and room acoustics. Varying the reflectance function of the room acoustic rendering equation, different geometrical room acoustic models, such as the *image source method*, the *ray-tracing method*, the *radiosity method*, and the *sonel mapping method* have been derived. In section 4.2 a more extended derivation analysis is presented with details about functions.

4.2. Relation to acoustic radiative transfer equation

As mentioned above the radiative transfer equation represents a radiant energy balance equation in an infinitesimal volume element. The rendering equation method is also based on a radiation balance equation. However, in contrast to the acoustic radiative transfer equation, the room acoustic rendering equation balances incoming and outgoing radiation fluxes on infinitesimal surface elements [42]. Another important difference between them is that the acoustic radiative transfer equation includes in the propagation medium both absorption and scattering, whereas the room acoustic rendering equation does not inherently take into account the scattering media. However, it should be noted that scattering objects might be artificially modeled as room surfaces or boundaries. Otherwise, absorption that causes sound attenuation due to air particles is taken into account, and thus the absorption term in equation (16) is $\mu_a = m$.

Because of the differences mentioned above, let us begin deriving the room acoustic rendering equation starting from acoustic radiative transfer equation, equation (16),

[1] Some authors call this term the *bidirectional reflectance distribution functions* [9].

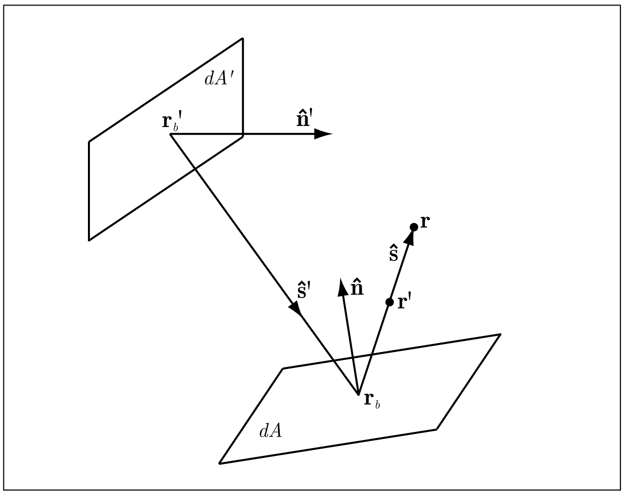

Figure 5. Distance between points $\mathbf{r}$ and $\mathbf{r}_b$.

ignoring the integral scattering term, but with the air absorption term included,

$$\frac{1}{c}\frac{\partial L(\mathbf{r},\hat{\mathbf{s}},t)}{\partial t} + \hat{\mathbf{s}}\cdot\nabla L(\mathbf{r},\hat{\mathbf{s}},t) = - mL(\mathbf{r},\hat{\mathbf{s}},t) + q(\mathbf{r},\hat{\mathbf{s}},t). \quad (44)$$

This is an inhomogeneous first order partial differential equation. It can be interpreted as a model for the sound energy propagation into an empty enclosure. Our goal is to obtain the outgoing sound radiance leaving a point on a surface $\mathbf{r}_b$ as a combination of the emitted radiance and, the amount of the reflected radiance incoming from all visible surfaces.

The solution to this equation is composed by the solution to the homogeneous form, and a particular solution. The general solution of the homogeneous first order partial differential equation is given by the following expression,

$$L_H(\mathbf{r},\hat{\mathbf{s}},t) = \mathrm{e}^{-m(\mathbf{r}-\mathbf{r}')\cdot\hat{\mathbf{s}}}\phi\left[t - \frac{(\mathbf{r}-\mathbf{r}')\cdot\hat{\mathbf{s}}}{c}\right], \quad (45)$$

where $\phi[u]$ is an arbitrary function and $\mathbf{r}'$ is a reference point. This equation follows the same expression as a plane wave that propagates in direction $\hat{\mathbf{s}}$ from $\mathbf{r}'$ to $\mathbf{r}$ and is attenuated along its path by the air attenuation factor m over distance $(\mathbf{r}-\mathbf{r}')\cdot\hat{\mathbf{s}}$. From equation (45) it can be seen that $\phi[u]$ also represents energy propagation. This function expresses the radiance at a delayed time $(\mathbf{r}-\mathbf{r}')\cdot\hat{\mathbf{s}}/c$. Therefore, since it is a plane wave, the radiance has to be evaluated at position $\mathbf{r}'$,

$$L_H(\mathbf{r},\hat{\mathbf{s}},t) = \mathrm{e}^{-m(\mathbf{r}-\mathbf{r}')\cdot\hat{\mathbf{s}}}L\left(\mathbf{r}',\hat{\mathbf{s}},t - \frac{(\mathbf{r}-\mathbf{r}')\cdot\hat{\mathbf{s}}}{c}\right). \quad (46)$$

Note that plane propagating waves are usually described in geometrical acoustic by rays representing the sound energy propagation in the direction of that plane wave.

To solve this Cauchy-problem subject to initial conditions let $\mathbf{r}'$ lie on the boundary $\mathbf{r}_b$, then the acoustic radiative transfer boundary conditions provide an expression for $L(\mathbf{r}',\hat{\mathbf{s}},t)$.

Substituting equation (18) into equation (46) yields

$$L(\mathbf{r},\hat{\mathbf{s}},t) = \mathrm{e}^{-m(\mathbf{r}-\mathbf{r}_b)\cdot\hat{\mathbf{s}}}\Bigg[L_0\left(\mathbf{r}_b,\hat{\mathbf{s}},t-\frac{(\mathbf{r}-\mathbf{r}_b)\cdot\hat{\mathbf{s}}}{c}\right) + \int_{\Omega-} R_F(\mathbf{r}_b;\hat{\mathbf{s}}',\hat{\mathbf{s}})L\left(\mathbf{r}_b,\hat{\mathbf{s}}',t-\frac{(\mathbf{r}-\mathbf{r}_b)\cdot\hat{\mathbf{s}}}{c}\right)(\hat{\mathbf{s}}'\cdot-\hat{\mathbf{n}})\Bigg]\,\mathrm{d}\Omega'. \quad (47)$$

using the notation expressed in Figure 5.

This equation is parameterized by direction as the integration term shows. In room acoustics the integration term is usually parameterized by position, changing the infinitesimal solid angle by a infinitesimal area surface along the incoming radiance from all visible boundary points $\mathbf{r}_b'$. Moreover, sound radiance at these points $L(\mathbf{r}_b',\hat{\mathbf{s}}',t)$ should be defined from equation (46),

$$\begin{aligned} L(\mathbf{r},\hat{\mathbf{s}},t) \;=\; &\mathrm{e}^{-m(\mathbf{r}-\mathbf{r}_b)\cdot\hat{\mathbf{s}}}\Bigg[L_0\left(\mathbf{r}_b,\hat{\mathbf{s}},t-\frac{(\mathbf{r}-\mathbf{r}_b\cdot\hat{\mathbf{s}})}{c}\right) \\ &+ \int_{\partial V} R_F(\mathbf{r}_b;\hat{\mathbf{s}}',\hat{\mathbf{s}})\,\mathrm{e}^{-m(\mathbf{r}_b-\mathbf{r}_b')\cdot\hat{\mathbf{s}}'} \\ &L\left(\mathbf{r}_b',\hat{\mathbf{s}}',t-\frac{(\mathbf{r}-\mathbf{r}_b)\cdot\hat{\mathbf{s}}+(\mathbf{r}_b-\mathbf{r}_b')\cdot\hat{\mathbf{s}}'}{c}\right) \\ &\cdot g(\mathbf{r}_b,\mathbf{r}_b')\,\mathrm{d}A'\Bigg], \end{aligned} \quad (48)$$

where $g(\mathbf{r}_b,\mathbf{r}_b')$ is the geometrical term defined as follows,

$$\mathrm{d}\Omega' = \mathrm{d}A'\frac{(\hat{\mathbf{s}}'\cdot\hat{\mathbf{n}}')}{\left((\mathbf{r}_b-\mathbf{r}_b')\cdot\hat{\mathbf{s}}'\right)^2}, \quad (49)$$

$$g(\mathbf{r}_b,\mathbf{r}_b') = \frac{(\hat{\mathbf{s}}'\cdot\hat{\mathbf{n}}')(\hat{\mathbf{s}}'\cdot-\hat{\mathbf{n}})}{\left((\mathbf{r}_b-\mathbf{r}_b')\cdot\hat{\mathbf{s}}'\right)^2}. \quad (50)$$

Usually, an artificial visibility term $v(\mathbf{r}_b,\mathbf{r}_b')$ is incorporated in the equation to ensure that boundary points are reachable and there are no obstacles between them.

Here the source term $L_0(\mathbf{r}_b,\hat{\mathbf{s}},t)$ is the sound emittance. A boundary surface can be considered as an emitter in the modeling of such processes as sound energy transmission through a room boundary or the primary reflected radiance of a point source $q(\mathbf{r},\hat{\mathbf{s}},t)$ with a direct contribution coming from inside of the room, not from another boundary position.

In order to obtain a complete solution, the direct sound contribution from a point source in the room should be incorporated in equation (48). This can be done through the particular solution defined by

$$L_P(\mathbf{r},\hat{\mathbf{s}},t) = \int_{-\infty}^{t} q(\mathbf{r}_s,\hat{\mathbf{s}},\tau)\,G(\mathbf{r}-\mathbf{r}_s,\hat{\mathbf{s}},t-\tau)\,\mathrm{d}\tau, \quad (51)$$

where $\mathbf{r}_s$ is the source position and $G(\mathbf{r},\hat{\mathbf{s}},t)$ is the free-field Green's function associated with equation (44). The free-field Green's function of acoustic radiative transfer equation is

$$G(\mathbf{r},\hat{\mathbf{s}},t) = \frac{\mathrm{e}^{-m(\mathbf{r}\cdot\hat{\mathbf{s}})}}{4\pi(\mathbf{r}\cdot\hat{\mathbf{s}})^2}\delta\left(t-\frac{\mathbf{r}\cdot\hat{\mathbf{s}}}{c}\right), \quad (52)$$

where $\delta(t)$ is the dirac delta function. This leads to

$$L_P(\mathbf{r},\hat{\mathbf{s}},t) = \frac{e^{-m(\mathbf{r}-\mathbf{r}_s)\cdot\hat{\mathbf{s}}}}{4\pi((\mathbf{r}-\mathbf{r}_s)\cdot\hat{\mathbf{s}})^2} \cdot q\left(\mathbf{r}_s,\hat{\mathbf{s}},t-\frac{(\mathbf{r}-\mathbf{r}_s)\cdot\hat{\mathbf{s}}}{c}\right). \quad (53)$$

The last equation represents the direct sound energy from a point source inside the room that arrives after some time with a spherical divergence and attenuation due to air absorption. The combination of equation (48) and equation (53) is a solution that makes it possible to calculate the sound radiance at any point in space propagating at $\hat{\mathbf{s}}$ direction as a result of the sum of the direct and reflected sound energy.

The complete room acoustic rendering equation incorporating a time-dependence parameter is a particularization of equation (48) when a boundary point $\mathbf{r}_b$ is calculated. Although this derivation confirms the room acoustic rendering equation proposal of [9], it should be noted that the approach of this paper extends details about the implementation of this equation, such as time delays and direct sound. The reflection kernel term in equation (43) becomes

$$R(\mathbf{r}_b;\hat{\mathbf{s}}',\hat{\mathbf{s}}) = e^{-m(\mathbf{r}_b-\mathbf{r}'_b)\cdot\hat{\mathbf{s}}'} R_F(\mathbf{r}_b;\hat{\mathbf{s}}',\hat{\mathbf{s}}) \cdot g(\mathbf{r}_b,\mathbf{r}'_b)\, v(\mathbf{r}_b,\mathbf{r}'_b). \quad (54)$$

Since the room acoustic rendering equation models only the energy exchange between surface points, the sound radiance arriving from all visible surfaces and the direct energy arriving from a point source need to be evaluated separately in order to obtain the total sound radiance at a receiver point within the room. However, with the equation (48), it can be obtained directly.

After this derivation, the room acoustic rendering equation can be considered a particularization of the acoustic radiative transfer equation for empty enclosures.

5. Assumptions, advantages and disadvantages

The basic assumptions of the acoustic radiative transfer model (see section 2.1) are the same as those of geometrical acoustics where the wavelengths are regarded as short compared with the room dimensions and interference between rays is neglected [43]. Since no phase relations are taken into account, it is important to note that the Schroeder's frequency limit must be considered [44]. This restriction limits the validity of the acoustic radiative transfer equation to medium and high frequencies.

The main contribution of the acoustic radiative transfer model is that it expands classical geometrical room acoustic modeling algorithms incorporating a propagation medium and objects within the room that can absorb (see equation (6)) and scatter energy (see equation (5)).

Moreover, the model can handle complex sound sources and arbitrary reflections both on boundaries and in the medium. The scattering on the boundaries uses the reflection function (see section 2.4) that represents the directional distribution of the reflected energy for each incoming angle. In the case of the scattering medium, the phase function is equivalent to the reflection function (see equation (14)).

Absorption and scattering of the medium are important in huge rooms and fitted rooms [19]; accordingly this model can predict realistic environments where modes and interference effects are not essential for our perception of sound. The scattering medium approximation could be useful in rooms where the interior contains noticeable objects (fittings), e.g. machines, chairs, and desks. Some factories, classrooms, and offices are studied as fitted rooms [45]. In these cases, the theory for empty rooms no longer holds - the fittings inside the room should be considered. Although these properties of the medium are defined with statistical terms that simplify the model because air conditions in rooms are usually isotropic, it is possible to expand the model to spatially dependent medium properties. It is important to note that the absorption and scattering coefficient are frequency dependent; therefore sound radiance propagation may be separated into different frequency subbands, which implies that a calculation must be carried for each frequency band. Although in this paper a room acoustic simulation model has been developed, it should be emphasized that it is used in this first approach more as a theoretical model than a simulation model in order to establish and unify the principles of a number of room acoustic simulation models.

The high computational cost involved in the solution of the integro-differential equation in five independent variables is one of its limitations [3]. Nevertheless, analytical solutions to the radiative transfer equation exist for simple cases, but for media with complex multiple scattering effects numerical methods are required [39]. In order to evaluate the possibilities of this model in predicting sound fields in rooms, two particularizations have been examined leading to an analytical approximation, the acoustic diffusion model, and a simplified integral solution, the rendering equation model, both corresponding to previously published well-know methods. In addition to providing a common and proper foundation, this model links the acoustic diffusion equation model with geometrical acoustic models deriving each method with the same theoretical foundations and clarifying theirs assumptions and limitations.

The diffusion approximation needs particular assumptions in addition to the general assumptions mentioned above. First, it requires the assumption of directional broadening (see section 3.2), which means that the scattering density must be high and the reflection of energy must dominate over absorption. To achieve this condition, all boundaries are assumed totally diffusely reflecting, leading to an isotropic phase function in the radiative transfer equation (see equation (33)). Accordingly, the receptor must also be sufficiently far from the source and the boundaries in space and in time to secure that enough reflection events have occurred when they reach the receptor, making the acoustic diffusion model inherently not valid for the direct field, only for the reverberant field. However, the accuracy near the boundaries is improved with the boundary conditions exposed in section 3.3. Secondly, the assumption of temporal broadening (see section 3.2) must be fulfilled, which means that the sound energy flow vector and the sound energy density may be nearly isotropic after sufficient scattering, and hence their variation per mean free path must be low and the expression for the sound energy flow vector is analogous to Fick's law (see equation (19)). Therefore, the directional meaning of these variables is lost. The source must be also isotropic, implying that only omnidirectional sound sources $\mathbf{q}_1(\mathbf{r}, \hat{\mathbf{s}}, t) = 0$ are applicable.

Theoretically, the acoustic diffusion model is mainly valid for predicting the late part of decay curves for low absorption cases in rooms with high accuracy. The diffusion equation model can be seen as an acoustic geometrical method, similar to ray-tracing and image-source models. Studies in the technical literature have shown that the diffusion model is computationally efficient compared with ray-tracing model [22, 28]. The diffusion model also provides more satisfactory results than statistical room acoustic theory since it is capable of modeling the non-uniformity of sound fields [22, 24]. Note that anisotropic component of the radiance is used in the approximation (see section 3.2).

The room acoustic rendering equation has been shown to be applicable to predict room sound fields in realistic environment with reasonable results in a round robin test [9]. Moreover, based on this general equation, an acoustic radiance transfer method, both for diffuse and non-diffuse reflection, has been presented [46] and have been extended to model diffraction [47]. Its main limitation is that the scattering events in the medium are neglected (see equation 44). Scattering objects might be modeled as room surfaces individually with their own scattering coefficients, which could be computationally more demanding than using the statistical scattering medium model. Its prediction results are expected to be more accurate than the acoustic diffusion model values, because this equation is not an approximation, especially when the specular reflections are dominant, with high absorption cases or when the assumptions of the directional and temporal broadening assumptions are not met. However, this precision is achieved with higher computational costs.

6. Conclusions

An acoustic radiative transfer equation theoretical approach has been presented for modeling acoustic energy propagation. It relies on geometrical acoustics. It may serve as a basis for developing more general and accurate prediction methods for room acoustics.

In order to evaluate the possibilities of this model in predicting sound fields in rooms, two of the most relevant equation models that have recently appeared in the architectural acoustics field, the diffusion equation model and the room acoustic rendering equation, have been derived and shown to be special cases of this general model. Thus the acoustic radiative transfer model provides a common acoustic geometrical foundation to both models and establishes a direct link between them. Moreover, this model makes it possible to include the diffusion equation model in the group of geometrical acoustic methods.

Using the proposed propagation model, the advantages and limitations of the two methods have been reviewed. It has been shown that the acoustic diffusion model is mainly valid for predicting the late reverberation part of decay curves in rooms with low absorption. On the other hand, the room acoustic rendering equation cannot inherently handle scattering events within the room.

Systematic experiments should be carried out in order to a) analyze how reliable the acoustic radiative transfer equation is to be used directly in room acoustic simulations and b) compare in detail the advantages and disadvantages of both the diffusion equation and room acoustic rendering equation models, clarifying the relationship of these two models in terms of accuracy, computational cost and applicability. These issues are expected to be addressed in future work.

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