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Technical Note

Implementation and evaluation of a diffusion equation model based on finite difference schemes for sound field prediction in rooms

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ABSTRACT

In this paper, the use of finite difference schemes for the acoustic diffusion equation model is introduced. Their features and limitations are analysed to select the adequate scheme based on both the stability conditions and the error order. The air absorption effects on the implementation are also discussed in terms of stability. To investigate the validity of the implementation, a set of simulations was conducted in a cubic room with four different absorption distributions. This evaluation was done by increasing either the spatial or the temporal resolutions of the studied scheme. The predicted values are compared with the statistical theory and geometrical models. The simulations suggested an empirical criterion for predicting the spatial and temporal resolutions that maximise the performance of the finite difference scheme.

1. Introduction

Recently, a diffusion equation model has been successfully applied to predict room-acoustic parameters, such as the reverberation time and the sound pressure level, in different room scenarios [1–4]. The acoustic diffusion equation models the sound field in enclosures with diffusely reflecting boundaries with *sound particles* with the same constant energy propagating along straight lines. This model, which takes into account the total diffuse reflections, has been demonstrated to be mainly valid for predicting the late part of the sound fields [1,4,5].

In previous works, a finite element method commercial software has usually been applied to model a wide variety of room types [1,3,4,6]. In this paper, the implementation of this acoustic diffusion equation model using finite difference schemes is investigated, and the appropriate scheme is presented as an alternative method that can be easily implemented.

This paper reviews the possible finite difference schemes available in the literature to find the most suitable scheme to implement a numerical solution for the diffusion equation model. Special attention is paid to the inclusion of the air absorption effect into the scheme and the analysis of such effect on the overall stability condition. Then, two selected explicit schemes, forward-time centred-space (FTCS) and Dufort–Frankel (DF), are compared with each other. These schemes are evaluated in several cubic rooms with different absorption distributions, and the predicted values are compared with the data obtained from Barron's statistical theory and two geometrical models: radiosity and diffuse ray-tracing. Thus, this paper aims to evaluate the accuracy, the stability and the computational cost of a finite difference implementation of a diffusion equation model.

This paper is organised as follows. First, the acoustic diffusion equation model is reviewed. Then, several available finite difference schemes are investigated, and theirs stability conditions are derived with the air absorption included in the equation. Next, the implementation of the algorithm is evaluated with the chosen schemes. Finally, the conclusions are presented.

2. Acoustic diffusion equation model

The use of the diffusion equation in acoustics was first proposed by Ollendorff [7], and further developed by Picaut et al. [8]. The model is based on an analogy between the transport of sound particles in a room with diffusely reflecting boundaries and the diffusion of particles in a medium containing scattering objects [9]. Recently, an alternative derivation of this model has used an analogy with the radiative transfer theory for the transport of photon energy in a scattering medium [10].

To derive the model, two main assumptions over the acoustic radiative transfer model are required. The first assumption is the directional broadening, which implies that the scattering density must be high and that the energy reflection must dominate over

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the energy absorption. Hence, the sound energy density becomes nearly the same in all positions after numerous diffuse reflections. The second assumption is the temporal broadening, which states that in a primarily scattering medium, the fractional change in the sound energy flow vector over one transport mean free path is much less than unity. With these assumptions about the behaviour of sound particles in a scattering medium, a mathematical diffusion approximation is established [10].

The diffusion equation model for the sound energy density $w(\mathbf{r},t)$ at position $\mathbf{r}$ defined on a domain V and time t, which includes a sound source term $P(t)$ located at position $\mathbf{r}_s$, consists of a partial differential equation with mixed boundary conditions [1,11],

$$\frac{\partial w(\mathbf{r},t)}{\partial t} - D\nabla^2 w(\mathbf{r},t) + cmw(\mathbf{r},t) = P(t)\delta(\mathbf{r}-\mathbf{r}_s) \text{ in } V, \tag{1}$$

$$-D\frac{\partial w(\mathbf{r},t)}{\partial \mathbf{n}} = A_X(\mathbf{r},\alpha)cw(\mathbf{r},t) \text{ on } \partial V. \tag{2}$$

Eq. (1) is an inhomogeneous parabolic partial differential equation, where ∇^2 is the Laplace operator and $D = \lambda c/3$ is the so-called diffusion coefficient with c being the speed of sound. This diffusion coefficient takes into account the room geometry through its *mean free path* λ, which indicates the average distance that a sound particle travels between two consecutive collisions [9]. In the classical acoustic theory, the mean free path in a room is given by $\lambda = 4V/S_t$, with volume V and total interior area S_t. The term $cmw(\mathbf{r},t)$ accounts for the atmospheric attenuation within the room, where m is the absorption coefficient of air [12].

Eq. (2) is a mixed boundary condition that models the local effects on the sound field induced by different degrees of absorption on the surfaces. The term $\mathbf{n}$ represents the unity vector normal to the boundary surface. This boundary condition equation allows one to express the distribution of the surface absorption properties through the absorption factor $A_X = A_X(\mathbf{r},\alpha)$, where α is the absorption coefficient. Different definitions of A_X have been presented in the technical literature, each depending on the assumed physical theory. In accordance with the diffuse sound-field theory [13], the Sabine absorption factor was presented. However, the Sabine absorption factor has been found to be accurate only for the rooms with low absorption, i.e., those with mean effective absorption values, $\bar{\alpha}$, below 0.2 [1,14,15]. To improve the predictions in the rooms with higher absorption values, where Sabine's theory is no longer valid, the Eyring absorption factor has been defined [6,16]. It should be noted, however, that the directional broadening assumption implies a high amount of diffuse reflections, which cannot be fully assured in high absorption rooms. In other words, the Eyring absorption factor should be in the range $\bar{\alpha} \leqslant 0.5$ to be effective [16]. Another derivation of this boundary condition, called the modified absorption factor [10,11], is based on a boundary condition for light diffusion in media, which avoids the singularity within the Eyring factor when the absorption coefficient of one material is 1.0. Furthermore, the main advantage of this boundary condition is its better accuracy for the higher absorption coefficients up to $\bar{\alpha} \leqslant 0.7$ [11]. In this paper, the modified absorption factor is adopted to perform the simulations.

3. Finite difference schemes for the acoustic diffusion model

3.1. Finite difference schemes for the governing equation

The finite difference method is a numerical technique used to solve a differential equation over a given region subject to the specified boundary conditions, which are based on a finite difference approach of the involved derivatives of a partial differential equation. When the finite difference approach is used, the problem domain is discretised so that the values of the unknown dependent variable are considered only at a finite number of nodal points or cells instead of at every point over the region. A discretised function is defined as follows,

$$w(\mathbf{r},t) = w(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = w^n_{i,j,k}, \tag{3}$$

where the temporal index n, and the spatial indexes i, j and k have been introduced, together with the temporal discretisation Δt and spatial discretisations in the cartesian axis Δx, Δy and Δz. Latters are defined as the inverse of the temporal and spatial resolutions respectively. For example, low resolution implies large cell sizes and high resolution implies small cell sizes.

A straightforward finite difference approach to solve the acoustic diffusion equation that includes the atmospheric absorption factor is the forward-time centred-space (FTCS) scheme. This scheme uses a central difference for the second derivative in space and a forward difference for the first derivative in time [17].

FTCS is called the simple explicit form of finite difference approximation of the diffusion Eq. (1), because it involves only one unknown at each time level, which can be directly calculated when the neighbour values at the previous time level are available. The accuracy of this scheme is first-order in time and second-order in space. The error order is defined as $O[(\Delta t)] + O[(\Delta v)^2]$, where $v = [x,y,z]$ and $O[\cdot]$ is the remaining error associated with a specific finite difference, and the variable inside the brackets determines the upper boundary on the growth rate of that scheme.

The calculated stability criterion for an FTCS scheme, including the air absorption phenomenon, implies that the value of Δt should be restricted to (see details in reference [18])

$$\Delta t \leqslant \left[2D\sum_{v=[x,y,z]}\frac{1}{(\Delta v)^2} - \frac{cm}{2}\right]^{-1}. \tag{4}$$

This stability criterion implies that for a given Δv, the magnitude of Δt cannot exceed the limit imposed by Eq. (4).

It should be noted that compared to having no atmospheric attenuation effect, the stability condition is slightly more permissive, allowing the use of coarser time discretisation, i.e., a lower temporal resolution. Nevertheless, the air absorption m usually varies from 10^{-5} to 0.05 m^{-1}, corresponding to a frequency range of 50–16,000 Hz for an acoustic wave propagating in air at 20 °C and a 50% relative humidity. The air absorption is so small that the effect in the FTCS algorithm stability can be ignored [19]. However, it should be emphasised that the appearance of the variable D in the stability criterion in Eq. (4) introduces a direct dependency with the entire analysed volume, which means that for a given Δt, a bigger room allows a greater spatial discretisation size, and a smaller number of cells have to be calculated. Although it may seem beneficial for big room simulations, such as theatres and churches, this condition shows increasing temporal resolutions and the computation times as Δv decreases. Therefore, when a small Δv is required due to the room model or the simulation specifications, Δt becomes very small, and many iterations are required to evolve the system through time. Additionally, if larger time steps are used to decrease the simulation time, then the error of the numerical solution is increased. To overcome these limitations and to improve the prediction accuracy, some alternative finite difference schemes have been investigated to avoid any stability restrictions either to the size of the time step Δt or the cell size Δv with a higher error order.

Another alternative scheme could be a simple implicit method called backward-time centred-space, which is accurate to $O[(\Delta t)] + O[(\Delta v)^2]$. This scheme is similar to the FTCS scheme, but in this case, the resulting scheme becomes unconditionally stable [17]. To improve the accuracy while maintaining no restrictions on the size of the time step, Crank and Nicolson [20] developed an alternative implicit difference scheme from the simple one, which is second order accurate in both time and space $O[(\Delta t)^2] + O[(\Delta v)^2]$ and is unconditionally stable [21]. Instead of an explicit form, the two previous schemes use an implicit form because at each time level, a set of equations are to be solved simultaneously to determine the nodal function value. The major drawback with these schemes is solving the equations. In the case of 1D transient problems the tridiagonal matrix algorithm is an easy and fast mathematical solution; however, for the 2D and 3D problems, each point to be updated involves four or more neighbouring point values, and these implicit schemes have not an easy and fast solution to implement [22]. Alternatively, there are explicit scheme methods that, in trying to obtain an unconditionally stable solution, become either unconditionally unstable or inconsistent with the differential equation that they are intended to approximate. For example, Richardson [23] proposed a three-time-level scheme for a finite difference approximation of the diffusion equation, but unfortunately, this method is unconditionally unstable.

However, Dufort and Frankel (DF) [24] proposed an expansion of this scheme in order to stabilize it. The resulting scheme has a truncation error of order $O[(\Delta t)^2] + O[(\Delta v)^2] + O[(\Delta t)^2(\Delta v)^{-2}]$ and is unconditionally stable although it is explicit. This explicit and consistent scheme for parabolic equations is convergent only if $(\Delta t)^2(\Delta v)^{-2}$ tends to zero as Δt and Δv approach zero [25]. These attractive features in the DF scheme allow this method to be the most suitable to apply to the acoustic diffusion equation model.

Here, for the purposes of this paper, the 3D problem governed by the acoustic diffusion equation will be developed using DF scheme. For simplification, let us define $\beta_{0_v} = (2D\Delta t)/(\Delta v)^2$, and $\beta_0 = \sum_{v=[x,y,z]} \beta_{0_v}$, so the finite difference scheme is as follows,

$$\begin{aligned} w_{i,j,k}^{n+1}(1+\beta_0) = {} & w_{i,j,k}^{n-1}(1-\beta_0) - 2\Delta t c m w_{i,j,k}^{n} \\ & + \beta_{0_x}\left(w_{i+1,j,k}^{n} + w_{i-1,j,k}^{n}\right) \\ & + \beta_{0_y}\left(w_{i,j+1,k}^{n} + w_{i,j-1,k}^{n}\right) \\ & + \beta_{0_z}\left(w_{i,j,k+1}^{n} + w_{i,j,k-1}^{n}\right). \end{aligned} \tag{5}$$

The source term, included as a soft source, is added at the suitable position as $w_{i_s,j_s,k_s}^{n+1} = w_{i_s,j_s,k_s}^{n+1} + 2\Delta t P_{i_s,j_s,k_s}^{n}$. Despite the inherent simplicity of using a hard source, this does not correspond to any analytical solution for the partial differential equations [17].

This scheme is theoretically unconditionally stable [24]. However, to investigate the stability of the algorithm when the air absorption is included, Section 3.3 presents the derivation and the corresponding explanations.

3.2. Finite difference schemes for the boundary-value problem

Eq. (5) is called explicit form because it involves only one unknown $w_{i,j,k}^{n+1}$. Clearly, the system of equations obtained from Eq. (5), i.e., in dimension x, for $i = 1, 2, \ldots, L_{x-1}$, provides L_{x-1} algebraic relations but contains L_{x+1} unknown nodes at time step $n + 1$, for $i = 0, 1, 2, \ldots, L_x$. Additional relations are needed to make the number of equations equal to the number of unknown variables. These variables are obtained from boundary conditions, explained as follows.

For simplicity, let us adopt a boundary surface oriented on the x-axis at both positions $x = 0$ and $x = l_x$,

$$-D\frac{\partial w(\mathbf{x},t)}{\partial x} + cA_X(\mathbf{x},\alpha)w(\mathbf{x},t) = 0 \quad \text{at } x = 0, \tag{6}$$

$$D\frac{\partial w(\mathbf{x},t)}{\partial x} + cA_X(\mathbf{x},\alpha)w(\mathbf{x},t) = 0 \quad \text{at } x = l_x, \tag{7}$$

where the acoustical density energy at the boundary nodes $i = 0$ and $i = L_x$ are unknown. Additional relations are obtained by discretising these two boundary conditions.

First-order accuracy is directly calculated if forward finite difference scheme for Eq. (6) and backward finite difference scheme for Eq. (7) is applied. However, the results are only of first-order accuracy $O[(\Delta v)]$ [25]. Therefore, to ensure the accuracy of the approximation, a second-order accurate difference of the boundary conditions is desirable. The finite difference approximation of these boundary conditions at the nodes $i = 0$ and $i = L_x$ using the second-order accurate three-point formula of the first derivative at the boundary nodes gives:

$$\frac{D}{2\Delta x}\left(3w_{0,j,k}^{n+1} - 4w_{1,j,k}^{n+1} + w_{2,j,k}^{n+1}\right) + A_{X_{0,j,k}} w_{0,j,k}^{n+1} = 0, \tag{8}$$

$$\frac{D}{2\Delta x}\left(w_{L_x-2,j,k}^{n+1} - 4w_{L_x-1,j,k}^{n+1} + 3w_{L_x,j,k}^{n+1}\right) + A_{X_{L_x,j,k}} w_{L_x,j,k}^{n+1} = 0. \tag{9}$$

Then the following two expressions are derived, respectively,

$$w_{0,j,k}^{n+1} = \frac{4w_{1,j,k}^{n+1} - w_{2,j,k}^{n+1}}{3 + \frac{2A_{X_{0,j,k}}\Delta x}{D}}, \tag{10}$$

$$w_{L_x,j,k}^{n+1} = \frac{4w_{L_x-1,j,k}^{n+1} + w_{L_x-2,j,k}^{n+1}}{3 + \frac{2A_{X_{L_x,j,k}}\Delta x}{D}}. \tag{11}$$

The derivation for dimension y and dimension z is straightforward.

Thus, Eqs. (10) and (11), together with Eq. (5) for $i = 1, 2, \ldots, L_{x-1}$ are the complete finite difference approximations with the DF method of the acoustic diffusion equation in 3D subjected to mixed boundary conditions.

3.3. Von Neumann analysis

In this section a Von Neumann analysis is performed to find the stability condition of Eq. (5). This is done by assuming a numerical plane wave of the form $w(\mathbf{x},t) = \xi^{t/\Delta t} e^{-\imath \mathbf{k}_0^T \mathbf{x}}$, where ξ is the amplification parameter, T means the transposed vector and $\mathbf{k}_0 = [k_{0_x}, k_{0_y}, k_{0_z}]$ is the wave number vector. Thus, this derivation aims to find under which conditions the requisite $\|\xi\| \leqslant 1$ (or alternatively, $\|\xi\|^2 \leqslant 1$) is verified [17].

After applying the Von Neumann procedure to Eq. (5) and some basic algebraical simplifications, the following expression is obtained,

$$(1+\beta_0)\xi = (1-\beta_0)\xi^{-1} - 2\Delta t c m + \sum_{v=[x,y,z]} \beta_{0_v}\left(e^{-\imath k_{0_v}\Delta v} + e^{\imath k_{0_v}\Delta v}\right). \tag{12}$$

Therefore, Eq. (12) is equivalent to

$$(1+\beta_0)\xi^2 - \left[\sum_{v=[x,y,z]} 2\beta_{0_v}\cos(k_{0_v}\Delta v) - 2\Delta t c m\right]\xi + (\beta_0 - 1) = 0. \tag{13}$$

The roots of this second order polynomial are expressed in the following way,

$$\xi = \frac{\sum_{v=[x,y,z]} \beta_{0_v}\cos(k_{0_v}\Delta v) - \Delta t c m}{1+\beta_0} \pm \frac{\sqrt{(1-\beta_0^2) + \left[\sum_{v=[x,y,z]} \beta_{0_v}\cos(k_{0_v}\Delta v) - \Delta t c m\right]^2}}{1+\beta_0}. \tag{14}$$

Now, examining these roots in detail, two different cases can be found. In the first case, if $(1-\beta_0^2) + \left[\sum_{v=[x,y,z]} \beta_{0_v}\cos(k_{0_v}\Delta v) - \Delta t c m\right]^2$ has non-negative values, it yields,

$$\begin{aligned} \|\xi\| & \leqslant \frac{\beta_0 - \Delta t c m + \sqrt{(1-\beta_0^2) + [\beta_0 - \Delta t c m]^2}}{1+\beta_0} \\ & = \frac{\beta_0 - \Delta t c m + \sqrt{1 + (\Delta t c m)^2 - 2\beta_0 \Delta t c m}}{1+\beta_0}. \end{aligned} \tag{15}$$

Therefore, in order to ensure stability the following condition has to be met:

$$\frac{\beta_0 - \Delta tcm + \sqrt{1 + (\Delta tcm)^2 - 2\beta_0 \Delta tcm}}{1 + \beta_0} \leqslant 1, \tag{16}$$

which is equivalent to $\beta_0 \geqslant -1$. Since Δt and Δv are always strictly positive, that condition always holds.

In the second case, if $(1 - \beta_0^2) + \left[\sum_{v=[x,y,z]} \beta_{0_v} \cos(k_{0_v}\Delta v) - \Delta tcm\right]^2$ is negative, it is found that,

$$\begin{aligned} \|\xi\|^2 &= \frac{\left[\sum_{v=[x,y,z]} \beta_{0_v} \cos(k_{0_v}\Delta v) - \Delta tcm\right]^2 - 1}{(1+\beta_0)^2} \\ &\quad + \frac{\beta_0^2 - \left[\sum_{v=[x,y,z]} \beta_{0_v} \cos(k_{0_v}\Delta v) - \Delta tcm\right]^2}{(1+\beta_0)^2} \\ &\leqslant \frac{\beta_0^2 - 1}{(1+\beta_0)^2}, \end{aligned} \tag{17}$$

then, the stability condition is achieved if

$$\frac{\beta_0^2 - 1}{1 + \beta_0^2 + 2\beta_0} \leqslant 1, \tag{18}$$

condition which is unconditionally true.

Therefore, the DF algorithm to be used in the diffusion equation model that includes the air absorption is theoretically unconditionally stable.

4. Algorithm evaluation

Both the DF and FTCS algorithms are evaluated in a cubic room using four different absorption distributions on the walls. This geometry is chosen to investigate the convergence and the performance of the implementation. Moreover, the absorption distribution is varied to evaluate the behaviour and the consistency of the predicted results. The results are also compared with the acoustical diffuse ray-tracing and the radiosity predictions. Finally, the computational requirements are discussed. Air absorption was neglected throughout the experiment because it has no significant effect on the stability of the implementation when it is incorporated as an additional term in Eq. (1).

The room has an $8 \times 8 \times 8\,\text{m}^3$ volume and an average absorption coefficient of 1/6. The four different absorption distributions being used are the absorption coefficient $\alpha = 1/6$ over all of the walls (room *A*); $\alpha = 1$ on the floor and $\alpha = 0$ over the rest of the walls (room *B*); $\alpha = 1/2$ on the floor and the ceiling and $\alpha = 0$ over the rest of the walls (room *C*); and $\alpha = 1/2$ on the floor and the front wall, while $\alpha = 0$ over the rest of the walls (room *D*). Several runs were conducted for each distribution with various spatial resolutions and temporal resolutions. A source was located at the centre of the cube with a sound power of 5 mW, and the receiver point was situated inside the enclosure that was 2 m from each the floor, the front wall and a sidewall. Simulations were run on a PC with an Intel® Core™2 Quad CPU 2,5 GHz processor and 2 GB RAM using Matlab®.

The energy decay function obtained by the algorithm implementation was used to calculate five room-acoustic parameters: the reverberation time (RT), the early decay time (EDT), the definition (D_{50}), the clarity (C_{80}) and the centre time (TS), which will be useful to perform the analysis as the same procedure as Ref. [5] is followed. Due to the directional broadening assumption, an intrinsic feature of the diffusion equation model presents an exclusion by the model of both the direct sound and the discrete early reflections [5], making a diffusion equation model not reliable for calculating energetic balance factors such as definition or clarity. Because the technical literature has shown that the diffusion equation model results in linear energy decays and it is an extension of the theory of diffuse sound fields [8], these predictions could be assumed to agree with Barron's revisited theory [26], which is based on the idealised integrated impulse curves according to the Eyring reverberation time equation. Therefore, this information might be used as the expectation values and as a reference to check the plausibility of the diffusion equation model results [27]. It should be pointed out this type of estimation gives no exact results but only the order of magnitude of these room-acoustic parameters.

4.1. Spatial resolution

The discretisation of the enclosure is performed in terms of finite differences, which means that the entire room is divided into cubic cells whose side is equal to Δv in each Cartesian direction. For the specific problem of the diffusion equation, the temporal broadening assumption determines that the size of the cells should be on the order of (or smaller than) one mean free path. Thus, the diffusion model can be applied to very large enclosures with a limited meshing. Regarding the architectural details of the room, unless the details need to be simulated, e.g., a small door or an aperture between two coupled rooms, Δv has to be selected as the order of one mean free path.

The accuracy of the numerical solution is related to both time and space resolutions, as indicated by the error order of each scheme. If the temporal resolution is fixed, this discretisation error decreases as the meshing resolution is increased because the numerical solution approaches the analytical solution. Otherwise, it is desirable to minimise the number of cells used in the discretisation to reduce the computational requirements. The Dufort–Frankel scheme is of second-order error in time and space, so it is expected to converge faster than FTCS. As it has been discussed in Section 3.1, the FTCS scheme is of first-order error in time; therefore, a smaller time step (Δt) compared to DF is necessary to obtain a similar error. Moreover, a larger restriction of FTCS performance lies in its stability condition (see Eq. (4)): for a smaller cell, a higher temporal resolution is needed, which subsequently increases the number of iterations needed [17,25]. These issues seem to support the DF scheme as the most suitable choice to simulate the diffusion equation model; however, its most remarkable advantage is its theoretically unconditional stability, which allows one to choose the temporal and the spatial resolutions without restrictions.

To investigate these issues, the DF implementation of the acoustic diffusion equation model was run ten times for each cubic room with a decreasing cell size Δv, from 1.2 to 0.06 m. The temporal resolution was fixed to $(\Delta t)^{-1} = 64\,\text{ms}^{-1}$ to ensure that the absolute error is independent of it. The simulations were carried out over 1.0 s, and the estimated values were compared with the calculated data from Barron's revisited theory [27] with the influence of the direct field neglected. The obtained values using Barron's formulae are C_{80} = 1.81 dB, D_{50} = 43.86%, EDT = 1.20 s, RT = 1.20 s and TS = 86.63 ms.

In Table 1, the predicted values of the room-acoustic parameters of room *A* are presented for various cell sizes. It should be noted that the predicted values of each room-acoustic parameter converges to a finite value as the spatial resolution increases. To study the evolution of the results, the percentage differences between the predictions in consecutive Δv situations are evaluated. These differences decrease when the spatial resolution is increased. Among the different parameters, C_{80} is the slowest to converge; however, the percentage difference when $\Delta v \leqslant 0.5$ m is lower than 1% for all room-acoustic parameters. Indeed, in the first approach, a finer subdivision of enclosure is not necessary in these trials, and a cell size below 0.5 m appears to give acceptable predictions.

Table 1
Parameter predictions with DF scheme and their corresponding differences with reference model (indicated inside the brackets) with varying cell size for the cubic room *A* and $(\Delta t)^{-1}$ = 64 ms^{-1}.

Δv (m)	Number of cells	C_{80} (dB (dB))	D_{50} (% (%))	EDT (s (%))	RT (s (%))	TS (ms (ms))	$(\Delta t)^2(\Delta v)^{-2}10^{-8}$
1.2	296	4.21 (2.40)	56.66 (12.80)	0.90 (24.46)	0.91 (23.90)	61.36 (25.27)	0.01
1	512	2.95 (1.14)	50.35 (6.49)	1.06 (11.48)	1.06 (10.97)	73.56 (13.07)	0.02
0.8	1000	2.76 (0.95)	49.47 (5.61)	1.09 (9.03)	1.09 (8.42)	75.68 (10.94)	0.03
0.6	2370	2.90 (1.08)	50.48 (6.62)	1.08 (9.27)	1.09 (8.48)	74.27 (12.36)	0.06
0.5	4096	2.30 (0.48)	46.68 (2.84)	1.14 (4.92)	1.14 (4.58)	80.90 (5.73)	0.09
0.4	8000	2.17 (0.35)	45.92 (2.07)	1.15 (3.56)	1.16 (3.29)	82.44 (4.19)	0.15
0.2	64,000	1.92 (0.11)	44.52 (0.67)	1.19 (0.83)	1.19 (0.71)	85.39 (1.24)	0.61
0.1	512,000	1.80 (0.01)	43.88 (0.02)	1.20 (0.50)	1.20 (0.56)	86.80 (0.17)	2.44
0.08	1,000,000	1.78 (0.03)	43.75 (0.10)	1.20 (0.75)	1.20 (0.79)	87.04 (0.41)	3.81
0.06	2,370,371	1.79 (0.03)	43.77 (0.08)	1.20 (0.65)	1.20 (0.73)	87.05 (0.43)	6.78

The trends are highly similar for the other room scenarios. Each parameter considered converges to a finite value as the number of cells increases. To support this evidence, the error regarding Barron's reference model is also evaluated in terms of the *just noticeable differences* (JNDs) concept. According to ISO/DIS 3382 [28], the subjective threshold for the reverberation time, the early decay time and the definition index is 5%, whereas for the clarity index it is required to be 1 dB, and for the centre time is 10 ms (see Table 1). All of the parameters show values less than the subjective threshold for a cell size below 0.5 m, supporting the aforementioned findings.

4.2. Temporal resolution

Although the algorithm has been shown to always satisfy the stability criterion because it is unconditionally stable, further predictions were made to investigate the effect of the time resolution on the convergence and the accuracy of the numerical solution. For these simulations, seven temporal resolutions with increasing values from $(\Delta t)^{-1}$ = 1 ms^{-1} to 64 ms^{-1} for each Δv considered in the previous experiment (see Table 1) were used.

As occurred in the case with various Δv, the room-acoustic parameter predictions always converge to a fixed value when the time resolution increases. The estimated values of the parameters versus $(\Delta t)^{-1}$, using Δv = 0.4 m and Δv = 0.2 m to mesh the cubic room *A*, are showed in Tables 2 and 3, respectively. Again, the data inside the brackets are the difference between this model and the Barron's reference model. For Δv = 0.4 m, the consecutive simulation results have similar values with a percentage difference below 0.2%, when they are higher than $(\Delta t)^{-1}$ = 8 ms^{-1}. With this temporal resolution, the elapsed time was 3.96 s. The values presented for a spatial step of 0.2 m also follow the same convergence trend, resulting in a percentage difference for consecutive predictions below 0.2% when starting from 16 ms^{-1} in this case, in which the elapsed time was 54.52 s.

Again, the trends are similar for all room scenarios. As expected, each parameter converges to a finite value as the temporal resolution increases. Moreover, as seen in Tables 2 and 3, these predicted values were below the subjective difference limens specified in ISO/DIS 3382 compared to the results from the Barron's reference model. The error decreases as the temporal resolution is increased in all simulated configurations. It is of note that for each cell size, there is a time resolution value from which the error does not decrease even if the temporal resolution is increased.

This presented observational evidence combined with the authors' experience in many other simulations might suggest an empirical criterion to choose the spatial and time resolution: an order of magnitude of 10^{-8} for the relation $(\Delta t)^2(\Delta v)^{-2}$ ensures that the predictions converge to a fixed value with a very low error. Accordingly, to configure a simulation, a cell size is fixed, and a temporal resolution is calculated with this empirical criterion to obtain low error results with the minimum computational requirements.

4.3. Computational efficiency

The memory requirement depends exclusively on the spatial resolution needed for the particular simulation. The FTCS scheme needs to store only the neighbour node values at the previous time level to calculate the unknown value at the present time. Therefore, the memory consumption, solved by the mathematical derivation, is associated with a memory complexity of order 2*n*, where *n* is the total number of cells in which the model is discretised. Meanwhile, the DF scheme (see Eq. (5)) needs to store the value of the sound energy density in the two time steps before the one that is being calculated. In theory, this requirement results in a memory complexity of order 3*n*. However, it is possible to implement an algorithm to reduce the requirements to 2*n*, similar to FTCS, by using the same matrix w^{n-1} to store the calculated w^{n+1} values in each iteration [17]. Moreover, it is always possible to select a higher Δv for a fixed Δt using the DF scheme than the FTCS, which consumes less memory.

The elapsed time of the diffusion equation algorithm based on a finite difference scheme depends on the spatial and the temporal resolutions as well as the simulated length of the impulse response. As expected from above, in all of the simulations performed using both the FTCS and the DF implementations, a similar elapsed time was observed for both methods when the configuration parameters were the same. Fig. 1 shows the DF implementation elapsed time

Table 2
Parameter predictions using DF scheme and their corresponding differences with reference model (indicated inside the brackets) with varying $(\Delta t)^{-1}$ for the cubic room *A* and cell size Δv = 0.4 m.

$(\Delta t)^{-1}$ (ms^{-1})	C_{80} (dB (dB))	D_{50}(% (%))	EDT (s (%))	RT (s (%))	TS (ms (ms))	$(\Delta t)^2(\Delta v)^{-2}10^{-8}$
1	3.22 (1.40)	52.19 (8.33)	0.99 (17.45)	0.97 (19.07)	68.37 (18.26)	625
2	2.62 (0.81)	48.96 (5.11)	1.11 (6.85)	1.11 (6.72)	76.36 (10.26)	156.25
4	2.32 (0.50)	47.02 (3.16)	1.14 (4.39)	1.15 (4.12)	80.36 (6.26)	39.06
8	2.16 (0.35)	45.90 (2.04)	1.15 (3.79)	1.15 (3.50)	82.36 (4.27)	9.76
16	2.17 (0.35)	45.94 (2.09)	1.15 (3.62)	1.16 (3.34)	82.37 (4.26)	2.44
32	2.17 (0.35)	45.93 (2.07)	1.15 (3.57)	1.16 (3.30)	82.42 (4.21)	0.61
64	2.17 (0.35)	45.92 (2.07)	1.15 (3.56)	1.16 (3.29)	82.44 (4.19)	0.15

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Table 3
Parameter predictions using DF scheme and their corresponding differences with reference model (indicated inside the brackets) with varying $(\Delta t)^{-1}$ for the cubic room *A* and cell size $\Delta v = 0.2$ m.

$(\Delta t)^{-1}$ (ms^{-1})	C_{80} (dB (dB))	D_{50}(% (%))	EDT (s (%))	RT (s (%))	TS (ms (ms))	$(\Delta t)^2(\Delta v)^{-2}10^{-8}$
1	1.93 (1.62)	40.45 (8.49)	0.90 (24.52)	0.47 (24.71)	56.21 (30.42)	2500
2	2.97 (1.15)	48.70 (4.85)	1.02 (14.84)	1.00 (16.40)	71.71 (14.92)	625
4	2.38 (0.57)	47.51 (3.65)	1.15 (4.15)	1.15 (4.12)	79.45 (7.17)	156.25
8	2.08 (0.26)	45.63 (1.77)	1.18 (1.65)	1.18 (1.53)	83.33 (3.30)	39.06
16	1.92 (0.11)	44.54 (0.69)	1.18 (1.04)	1.18 (0.90)	85.27 (1.36)	9.76
32	1.92 (0.11)	44.54 (0.67)	1.18 (0.87)	1.19 (0.75)	85.34 (1.29)	2.44
64	1.92 (0.11)	44.52 (0.67)	1.19 (0.83)	1.19 (0.75)	85.39 (1.24)	0.61

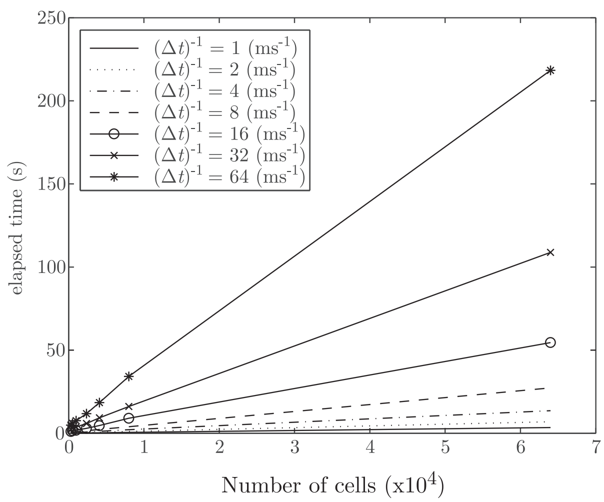

Fig. 1. Variation of elapsed time with number of cells for cubic room *A* with the Dufort–Frankel implementation.

versus the number of cells for different temporal resolutions with a response length of 1.0 s in room *A*. The run times for predictions for the other room scenarios are also similar.

However, the stability condition of FTCS (see Eq. (4)) is its main limitation, which increases the memory consumption and the elapsed time compared to DF. The computational efficiency comparison between FTCS and DF cannot be directly made using the same spatial and temporal resolution. If a cell size is fixed, the DF scheme can always use a temporal resolution lower than the FTCS scheme, resulting in a lower elapsed time. Moreover, as presented in Section 3.1, the error order of the DF scheme is higher than that of FTCS scheme, resulting in a lower error for the DF scheme predictions with the same temporal and spatial resolutions.

4.4. Comparison with other methods

In this section, the diffusion equation model results using both DF and FTCS are compared to the other well-known acoustic geometrical-based simulation methods, which also use diffuse reflections on the boundaries, such as the radiosity method and the diffuse ray-tracing method. Table 4 presents the results obtained from the aforementioned geometrical-based methods, which were obtained from Nosal et al. [29]. Table 4 also includes the predicted room-acoustic parameters using both the FTCS and the DF schemes with $\Delta v = 0.2$ m and their corresponding temporal resolution.

In all methods, the longest RT is in room *A*, where the absorption is uniform. The shortest RT is in room *B*, which has a high absorption factor at the floor near the receiver. The different RT values in each room indicates that the effect of different absorption distributions is well simulated by the implementations of the acoustic diffusion equation model. For higher RT, lower C_{80} and D_{50} values are obtained as expected, because a slower energy decay curve means more late energy, resulting in the lower early-to-late energy ratios. The TS predicted values are also consistent with the RT results.

The predicted values using FTCS and DF diffusion equation implementations are compared to the results from radiosity and diffuse ray-tracing in terms of JNDs. The FTCS predictions for rooms *A* and *D* are below the subjective difference limens as compared to both geometrical-based methods. However, differences over the limens are observables in all room-acoustic parameters for rooms *B* and *C* comparing with both geometrical-based methods. It is straightforward to state that the DF predictions for all room scenarios are below the subjective difference limens compared to both geometrical-based methods. In particular, acoustical radiosity gives quite similar predictions to the DF predictions. The parameter that exhibits the greatest difference between predictions is TS; however, most differences are below 10 ms. RT, D_{50} and EDT are all within 4%, and C_{80} is below 0.1 dB. Although similar results are obtained when the diffuse ray-tracing results are compared with the DF predicted values, these differences are higher than those obtained using radiosity method. These differences regarding geometrical-based methods lie in the different manners in which the diffuse reflection is incorporated in the models: the radiosity method and the acoustic diffusion equation model scatter rays in all directions, whereas random directions are used in diffuse ray-tracing [29]. These simulated results support the hypothesis that there are considerable differences between FTCS and DF implementations, which are associated with the different error orders of each scheme.

Table 4
Parameter predictions for the four cubic rooms using: (a) FTCS with $\Delta v = 0.2$ m and $(\Delta t)^{-1} = 96\ ms^{-1}$; (b) DF with $\Delta v = 0.2$ m and $(\Delta t)^{-1} = 8\ ms^{-1}$; (c) acoustic radiosity (RD) with 150 patches and $(\Delta t)^{-1} = 24\ ms^{-1}$ and (d) diffuse ray-tracing (DRT) with average over 64 runs with 1 million rays [diffuse reflection, 500 reflections, 24 kHz, $(0.1\ m)^3$ receiver].

	Room	C_{80} (dB)	D_{50} (%)	EDT (s)	RT (s)	TS (ms)		Room	C_{80} (dB)	D_{50} (%)	EDT (s)	RT (s)	TS (ms)
(a)	*A*	1.73	43.61	1.22	1.22	87.87	(c)	*A*	2.00	45.87	1.23	1.20	94.67
	B	5.89	63.07	0.70	0.70	50.35		*B*	2.85	49.88	1.08	1.06	84.50
	C	4.84	58.61	0.80	0.80	57.07		*C*	2.39	47.83	1.17	1.16	89.70
	D	1.80	43.99	1.21	1.21	87.00		*D*	2.35	47.58	1.17	1.16	90.20
(b)	*A*	2.08	45.63	1.18	1.18	85.33	(d)	*A*	2.02	45.87	1.22	1.23	94.50
	B	2.88	49.58	1.05	1.05	75.96		*B*	2.83	49.69	1.13	0.98	84.95
	C	2.31	46.45	1.12	1.12	82.28		*C*	2.31	46.24	1.18	1.16	91.38
	D	2.33	47.40	1.16	1.17	82.57		*D*	2.24	46.80	1.18	1.18	91.31

5. Conclusions

The implementation of the acoustic diffusion equation model for room-acoustic simulations using a finite difference scheme has been investigated. Both the forward-time centre-space (FTCS) and the Dufort–Frankel (DF) explicit finite difference schemes have been used, and their performance in cubic-shaped rooms have been evaluated.

Compared with the other schemes, the DF scheme has been shown to be the most suitable and simple solution, and it is easy to implement and to simulate with no stability restrictions and a second-order accuracy. The air absorption effect has been added to the implementation, and its stability condition has been derived so that the algorithm remains unconditionally stable and that the convergence of the algorithm is not affected.

Using both FTCS and DF implementations, several simulations have been carried out with different absorption distributions in a cubic room, and the room-acoustic parameters have been obtained. Simulated results give rise to a possible empirical criterion that can be used to calculate an empirical temporal resolution value from a fixed spatial resolution of the DF implementation, which helps one obtaining accurate predictions.

The comparisons to some geometrical-based methods, which also assume diffuse reflections, suggest that the DF implementation of the acoustic diffusion equation model obtains highly accurate predictions in this type of enclosure.

The computational complexity increases as the number of cells increases, whereas the memory complexity increases linearly with the doubling of the number of cells. Although both FTCS and DF implementations have the same memory and time complexities, the unconditionally stable feature of DF allows one to discretise the room with fewer cells than FTCS, resulting in a lower memory consumption and a decreased elapsed time.

It can be concluded that the Dufort–Frankel finite difference scheme for the diffusion equation model is a reliable method for performing accurate simulations with low computational complexity and calculation time. The present results suggest that a comparison between finite element and finite difference methods should be undertaken in future works.

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