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RESEARCH

Cooperative Location for Competing Firms under Delivered Pricing and Demand Linear in Price

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Abstract
The location-price decision problem for competing firms can be reduced to a location game if firms compete on delivered pricing. This game has often been studied for non-cooperative firms where the Nash equilibrium is used as solution concept. However, it may occurs that there exist alternative locations for which all firms get higher payoffs than those prescribed by the equilibrium. This fact has been shown by the authors for joint profit maximization locations when demand is fixed and firms set equilibrium prices. In this paper, we study the location game considering that demand is linear in price and firms cooperate by setting the monopoly price at each market instead of the equilibrium price. Two Mixed Integer Linear Programming location models are developed to maximize the joint profit for different and equal production costs, respectively. An empirical investigation is performed to compare the joint profit of the firms, which is obtained by the solutions of the proposed location models, with the one obtained by the Nash equilibrium solution that is obtained if the firms do not cooperate.

Keywords Facility location · Joint profit · Network optimization · Spatial competition

1 Introduction

Facility location and price decisions for competing firms is often studied as a two stage non-cooperative game. In the first stage, the competing firms select their facility locations, in the second stage, they compete on price with the aim of profit maximization. The division into two stages is motivated by the fact that the choice of

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location is a strategic decision usually prior to the decision on price. While location is a permanent decision, prices can be easily changed, and the firms will compete on price once the facility locations are fixed. The two-stage game can be reduced to a single stage location game if there exists a price equilibrium in the second stage, which is determined by the facility locations chosen in the first stage. The existence of price equilibrium often occurs when firms compete on delivered prices (see Pacheco 2009; Phlips 1983). With this price policy, firms offer a price at each market area, which includes production and transportation cost, and customers buy from the firm that offers the lowest price.

The notion of Nash equilibrium (*NE*) is probably the most used solution concept of the location game for noncooperative firms (Eichberger 1993; Tian 2015). Existence of a *NE* when firms compete on delivered pricing has been studied for spatially separated markets in both a transportation network and the plane as location spaces. If demand is fixed and marginal production cost is constant, it has been proved that any minimizer of social cost is a *NE* (see Dorta-González et al. 2005; Lederer and Hurter 1986; Lederer and Thisse 1990). On a network, there exists a global minimizer of the social cost at the nodes and the problem of minimizing the social cost has been solved when the firms locate multiple facilities by using Mixed Integer Linear Programming (*MILP*) formulations (see Pelegrín et al. 2011, 2017). On a plane, the problem of minimizing the social cost is harder to solve and it has been solved for two competing firms with single facility location (see Díaz-Bañez et al. 2011; Fernández et al. 2014). If demand is price-sensitive or marginal production costs are not constant, the socially optimal locations may not be a *NE* of the location game, as has been shown in Gupta (1994) and Hamilton et al. (1989). In such cases, to our knowledge, the existence of a pure strategy *NE* for the location game has not been proven yet. Although existence of a *NE* is not guaranteed for price sensitive demand, a *NE* can often be found when the location space is a network, as it is shown in Pelegrín et al. (2023).

An inconvenient of the *NE* solution is that there may exist a second solution such that the payoffs of each firm in the *NE* solution are lower than the corresponding payoffs in the second one (see Pelegrín et al. 2024). To avoid this inconvenient, the firms could cooperate with the aim of maximizing their joint profit and allow to earn higher payoffs than their respective *NE* payoffs. This kind of cooperation occurs in situations where members of an oligopoly, cartel, franchise system, or similar market condition engage in pricing-output decisions designed to maximize their profits as a whole (see for instance Alon et al. 2021; Kruse and Schenk 2000; Pelegrín et al. 2012; Thi Thao et al. 2022). In essence, the member firms seek to act as a monopoly and look for joint profit maximization facility locations (collusive solution of the location game). Although oligopoly collusion has extensively been studied in the economic literature, there is few research on how to solve the joint profit maximization location problem under delivered pricing. Most of the papers analyze incentives to collude and stability of collusion taking a linear market as location space (see for instance Andree et al. 2018; Gupta and Venkatu 2002; Huck et al. 2003; Matsumura and Matsushima 2005; Osborne and Rubinstein 1990; Thisse and Vives 1988).

Cooperation in network facility location under delivered pricing has been recently studied by the authors in Pelegrín et al. (2024) when demand is fixed and firms setting the equilibrium price in each market. If demand is not fixed, the equilibrium price is different from the monopoly price and the joint profit is maximized if each firm sets the monopoly price instead of the equilibrium price. Our contribution in this paper is to study cooperation for price sensitive demand with the agreement that firms will set the monopoly prices in their respective market areas. For delivered pricing, we show that the monopoly prices are determined by the facility locations, therefore joint profit maximization *(JPM)* becomes into a non-linear optimization problem with location variables. This problem is studied for the first time taking a network as location space. For demand linear in price we show that the problem can be formulated as a Mixed Integer Linear Programming (*MILP*) model for different and equal production costs, respectively. An empirical study is performed to test the proposed models and to compare the profits obtained by the competing firms if they select the monopoly prices to maximize the joint profit with the ones obtained if the firms do not cooperate and they select the equilibrium prices to find a *NE*.

The paper is organized as follows. In Section 2, the location game and the joint profit maximization problems are described. In Section 3, the computation of a *NE* for price sensitive demand is briefly discussed. In Section 4, we first show a node optimality property for the joint profit maximization problem with demand linear in price when firms set monopoly prices. Then two *MILP* formulations of (*JPM*) for different and equal marginal production costs are presented. In Section 5, the proposed models are tested by solving a variety of test problems and an experimental investigation is performed to compare the joint profit obtained with the *JPM* and the *NE* facility locations, respectively. Finally, some conclusions are considered in Section 6.

2 The Location Game and Joint Profit Maximization

We consider spatially separated markets which demand an homogeneous product at some nodes of a network $N = (V, E, l)$. The node set $V = \{v_k : k = 1, ..., n\}$ may contain nodes on which there is no market, which occurs if there are some linking nodes with no customer. Let v_k, $k = m + 1, ..., n$ the nodes with no demand. The edge set is $E = \{e : e = [v_k, v_j]; v_k, v_j \in V\}$, and $l(e)$ is the length of edge e. Distance between two points a and b in the network is measured as the length of the shortest path linking the two points and it will be denoted by $d(a, b)$ (see Pelegrín et al. 2012). There is a set of firms which compete for demand in the network. The firms have to make decisions on facility location and price with the aim of profit maximization. First, firms select the location of their facilities, then firms set delivered prices at each market. Note that location decision is permanent (strategic decision) while decision on price can be easily changed (tactic decision). It is assumed that the profit of any firm in any market is independent of the profit obtained in any other market. The marginal delivered cost is independent of the quantity delivered.

The following notation will be used:

Indices

i index of firms; $i = 1, \dots, r$.

j index of location candidates (in discrete location space); $j = 1, \dots, n$.

k index of demand nodes; $k = 1, \dots, m$.

Data

$F = \{1, 2, ..., r\}$	set of firms.
L	set of facility location candidates.
$M = \{1, 2, ..., m\}$	set of markets.
$d(x, k)$	distance between location x and demand node v_k; $x \in L, k \in M$.
$q_k(p)$	demand in market k as a function of the price p; $k \in M$.
f^i	number of facilities of firm i.
$pc^i(x)$	unit production cost of firm i at location x; $x \in L$.
T_i	a non-negative non-decreasing function; $i \in F$.
$tc^i(x, k) = T_i(d(x, k))$	unit transportation cost of firm i from location x to market k; $x \in L, k \in M$.
$dc^i(x, k) = pc^i(x) + tc^i(x, k)$	unit delivered cost of firm i from location x to market k; $x \in L, k \in M$.

Decision variables

X^i set of facility locations of firm i.

Miscellaneous

$X = (X^1, X^2, ..., X^r)$	location strategy for the facilities of the competing firms.

X^{-i}	vector of locations for the facilities of the firms other than i; $(X^i; X^{-i}) = X$.
$C_k(A) = \min\{dc^i(x,k) \,:\, x \in A\}$	minimum delivered cost from the facilities in A to market k.

2.1 The Location Game

Once facility locations are fixed, the firms will compete on price. Each firm offer a price at each market, which include production and transportation costs, then customers buy from the firm which offers the lowest price. It is assumed that the firms will not set a price below the marginal delivered cost. If two firms offer a minimum price at market k, the one with the minimum marginal delivered cost can lower its price and it obtains all the demand at market k. Then ties in price are broken in favour of the firm with the lowest marginal delivered cost. Therefore, we assume that each market will be served by the firms with the minimum marginal delivered cost. If more than one firm can price the minimum delivered cost at a given market, the competition process will make the profit obtained from that market to be zero.

Given the set of facility locations X, the prices at any market k are obtained as follows:

(i) If $C_k(X^i) < C_k(X^{-i})$, firm i obtains a maximum profit from market k by offering a price equal to the optimal solution of the following problem:

$$Max\{\Pi_k^i(p) = q_k(p)(p - C_k(X^i)) \,:\, C_k(X^i) \le p \le C_k(X^{-i})\}$$

The optimal solution to this problem is:

$$\hat{p}_k^i(X) = \begin{cases} p_k^{mon}(C_k(X^i)) & if\ p_k^{mon}(C_k(X^i)) \le C_k(X^{-i}) \\ C_k(X^{-i}) & if\ p_k^{mon}(C_k(X^i)) > C_k(X^{-i}) \end{cases}$$

where $p_k^{mon}(C_k(X^i))$ is the monopoly price of firm i at market k.

The monopoly price is the optimal solution to the following optimization problem:

$$Max\{\Pi_k(p) = q_k(p)(p - C_k(X^i)) \,:\, C_k(X^i) \le p \le p_k^{max}\}$$

where p_k^{max} is the maximum price that customers are willing to pay for the product at market k.

(ii) If $C_k(X^i) \ge C_k(X^{-i})$, then firm i obtains zero profit from market k. In this case, firm i sets a price $\hat{p}_k^i(X) = C_k(X^i)$ to make its competitors obtain a minimum profit from node k.

It is easy to see that $\hat{p}_k^i(X), i = 1, 2, \ldots, r$, are equilibrium prices at market k. We assume that for any strategy location X, the firms will set the equilibrium prices. Since the group of markets monopolized by each firm i is:

$$M^i(X) = \{k : C_k(X^i) < C_k(X^{-i})\}$$

the profit obtained by firm i is given by:

$$\Pi^i(X) = \sum_{k \in M^i(X)} q_k(\hat{p}^i_k(X))(\hat{p}^i_k(X) - C_k(X^i))$$

Then the location-price decision problem for the competing firms can be seen as a location game. The firms are the players, the alternatives of player i are subsets X^i of the set of location candidates L, and the payoff that player i obtains is $\Pi^i(X)$, $i = 1, ..., r$.

2.2 Joint Profit Maximization

It is well known that a *NE* may be an inefficient solution, i.e. there might exists a solution $\hat{X}$ such that $\Pi^i(\hat{X}) \geq \Pi^i(X)$ for any $i \in F$ and $\Pi^h(\hat{X}) > \Pi^h(X)$ for at least one $h \in F$ (see Pelegrín et al. 2024). In such a case, if the firms choose $\hat{X}$ as their facility locations they would obtain a joint payoff higher than the one prescribed by the *NE*. Then, as alternative to *NE*, the firms might cooperate with the aim of maximizing their joint profit. Note that a location strategy X which is a maximizer of the joint profit is also an efficient solution.

Once facility locations are fixed, the joint profit is maximized if each market is served by the facility with the minimum delivered cost and the monopoly price is set (cooperative solution). If more than one firm can price the minimum delivered cost in a given market, the profit obtained from that market will be shared by the firms with the minimum delivered cost. Note that in such a case the profit obtained from that market is not zero, contrary to what occurs if the firms set equilibrium prices (non cooperative solution).

If the firms accept facility locations maximizing their joint profit, they will face to the following non-linear optimization problem:

$$(JPM): \quad Maximize \quad \{\Pi(X) = \sum\nolimits_{k=1}^{m} q_k(p^{mon}_k(C_k(X)))(p^{mon}_k(C_k(X)) - C_k(X)) : \\ X = (X^1, X^2, ..., X^r), | X^i | = f^i, X^i \subset L, i = 1, .., r\}$$

where $p^{mon}_k(C_k(X)))$ is the optimal solution to the following optimization problem:

$$Max\{\Pi_k(p) = q_k(p)(p - C_k(X)) : C_k(X) \leq p \leq p^{max}_k\}$$

We call a *JPM* solution to any set of facility locations that maximizes the joint profit. This optimization problem is hard to solve in a non-linear location space. However, if demand is linear in price and the location space is a network, we will show that the problem can be solved by some formulations as *MILP* problem.

How to split the maximum joint profit among the firms is a point to be discussed. If each firm captures the profit from the markets it serves, two scenarios are possible. The first one is that the *JPM* solution is also a *NE* of the location game. Then the markets each firm serves under the *NE* solution are the same as the ones under the *JPM* solution. Since the profit from each market by setting the monopoly price is greater than the profit by setting the equilibrium price, it follows that the individual profit that each firms obtains when cooperating is always greater than without cooperating. The second scenario is that the *JPM* solution is not a *NE* of the location game, then one of the firms might get an individual profit under a *NE* solution greater than the individual profit obtained under the *JPM* solution. Then that firm might not cooperate which would suggest to split the joint profit in a different way.

An example of the second scenario is shown in Fig. 1, where the numbers on the edges are the unit transportation costs and the demand functions at the nodes are $q_1(p) = 9 - p$, $q_2(p) = 8 - p$ and $q_1(p) = 10 - p$. Two firms are considered, firm 1 with unit production cost $c_1 = 0.1$ and firm 2 with unit production cost $c_2 = 0$. The *JPM* solution is $X = (1, 3)$ with joint profit $\Pi(1, 3) = 48.61$ and individual profits $\Pi^1(1, 3) = 19.80$ and $\Pi^2(1, 3) = 28.81$ if each firm captures the profit from the markets it serves. The *NE* solution is $Y = (2, 3)$ with joint profit $\Pi(2, 3) = 45.36$ and individual profits $\Pi^1(2, 3) = 21$ and $\Pi^2(2, 3) = 24.36$. Note that firm 1 can get a profit greater than 19.80 by changing its location in the *JPM* solution from node 1 to node 2. Since $\Pi(1, 3) > \Pi(2, 3)$, to avoid that firm 1 change its location in the *JPM* solution, the joint profit $\Pi(1, 3)$ could be splitted among firm 1 and firm 2 in a different way so that each firm gets an individual profit greater than the one they got under the *NE* solution.

3 On the Computation of NE

The most used solution concept in location games for non cooperative firms is the Nash equilibrium. A strategic profile of locations $\hat{X} = (\hat{X}_1, \hat{X}_2, \dots, \hat{X}_r)$ is a *NE* if for any i it is verified that:

$$\Pi^i(\hat{X}^i, \hat{X}^{-i}) \geq \Pi^i(X^i, \hat{X}^{-i}), \forall X^i \subset L$$

To our knowledge, the only result that guarantees existence of a *NE* for the location game is in the case of fixed demand. In such a case a *NE* exists and can be found by minimizing the social cost (see Dorta-González et al. 2005; Pelegrín et al. 2011). Although existence of a *NE* has not been proved for price sensitive demand, finding a *NE* could be done by using the best response procedure if such equilibrium exists

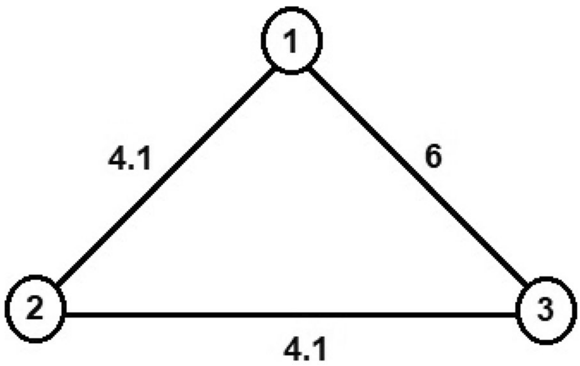

Fig. 1 Illustrative example

as it is shown in Pelegrín et al. (2023). The best location strategy for firm i, assuming that facility location of its competitors are known, is obtained by solving the following problem:

$$BR^i(X^{-i}) : \quad Max\{\Pi^i(Y^i, X^{-i}) : |Y^i| = f^i, Y^i \subset L\}$$

where X^{-i} are the facility locations for the firms other than i, $i = 1, .., r$. Based on the previous problem, the best response procedure is as follows:

1. Select an initial set of facility locations X_0. $X_0 = (X_0^1, X_0^2, ..., X_0^r)$, $|X_0^i| = f^i$, $i = 1, ..., r$. Set $v = 0$.
2. **For** $i = 1, ..., r$ **do** Find an optimal solution X_{v+1}^i to problem $BR^i(X_{v+1}^1, ..., X_{v+1}^{i-1}, X_v^{i+1}, ..., X_v^r)$. Set $X_{v+1}^i = X_v^i$ if $\Pi^i(X_{v+1}^1, ..., X_{v+1}^i, X_v^{i+1}, ..., X_v^r) = \Pi^i(X_{v+1}^1, ..., X_{v+1}^{i-1}, X_v^i, ..., X_v^r)$. **end for**
3. If $X_{v+1}^i = X_v^i$, $i = 1, ..., r$, then $X = (X_v^1, X_v^2, ..., X_v^r)$ is a *NE*, **STOP**. Otherwise, set $v = v + 1$ and **go to** step 2.

The previous procedure requires to solve problem $BR^i(X^{-i})$ many times. Then it can be used whenever problem $BR^i(X^{-i})$ is solved in appropriated run time. If L is not finite, the problem seems to be very difficult to solve, particularly for multi-facility location. If L is a finite set, the following Mixed Integer Linear Programming formulation can be used to solve $BR^i(X^{-i})$.

For $L = \{1, 2, \ldots, n\}$, let us define the following sets and variables:

$$L_k^i = \{j \in L : dc^i(j,k) < C_k(X^{-i})\}$$
$$L^i = \cup\{L_k^i : k \in M\}$$
$$M^i = \{k : L_k^i \neq \emptyset\}$$
$$x_j^i = \begin{cases} 1 & \text{if firm } i \text{ locates a facility at } j \\ 0 & \text{otherwise} \end{cases}$$
$$w_{kj}^i = \begin{cases} 1 & \text{if maket } k \text{ is served by firm } i \text{ from location } j \\ 0 & \text{otherwise} \end{cases}$$

L_k^i is the set of locations at which firm i can price below its competitors at market k and get the full demand at k. L^i is the set of locations at which firm i can get a positive profit. M^i is the set of markets where firm i can get a positive profit. Variables x_j^i and w_{kj}^i are location and allocation variables, respectively.

If market k is served from location $j \in L_k^i$, the equilibrium price at k is:

$$\hat{p}_k^i(j) = \begin{cases} p_k^{mon}(C_k(j)) & if\ p_k^{mon}(C_k(j)) \le C_k(X^{-i}) \\ C_k(X^{-i}) & if\ p_k^{mon}(C_k(j)) > C_k(X^{-i}) \end{cases}$$

Then the profit maximization problem for firm i can be formulated as follows:

$$BR^i(X^{-i}) : \quad \max \quad \sum_{k \in M^i} \sum_{j \in L_k^i} q_k(p_k^{mon}(C_k(j)) \quad (p_k^{mon}(C_k(j)) - C_k(j))w_{kj}^i$$

$$\text{s.t.} \quad \sum_{j \in L^i} x^i_j = f^i \tag{1}$$

$$w^i_{kj} \le x^i_j, \quad j \in L^i_k, \quad k \in M^i \tag{2}$$

$$\sum_{j \in L^i_k} w^i_{kj} \le 1, \quad k \in M^i \tag{3}$$

$$x^i_j, w^i_{kj} \in \{0, 1\}, \ j \in L^i_k, \quad k \in M^i \tag{4}$$

The objective function of problem $BR^i(X^{-i})$ represents the profit of firm i. Constraint (1) represents the number of facilities to be located by firm i. Constraints (2) guarantee that a variable w^i_{kj} may be positive only if firm i locates a facility at location j. Constraints (3) mean that each market $k \in M^i$ can be served from at most one of the facilities of firm i (the facility with the minimum marginal delivered cost in the optimal solution). Constraints (4) require that the variables are binary. The above problem is a Binary Integer Linear Programming (*BILP*) problem which contains a lot of binary variables. Observe that the optimal value of problem $BR^i(X^{-i})$ does not change if variables w^i_{kj} are taken as non negative variables instead of a binary variables. Replacing constraints $w^i_{kj} \in \{0, 1\}$ by $w^i_{kj} \ge 0$ in the above formulation, we obtain an equivalent problem which is a Mixed Integer Linear Programming (*MILP*) problem that let solve larger size problems.

4 JPM for Demand Linear in Price

Once X is fixed, for a linear demand function $q_k(p) = \alpha_k - \beta_k p$, the *monopoly price* in market k is the optimal solution to the problem:

$$\max \ \{q_k(p)(p - C_k(X)) \ : \ C_k(X) \le \frac{\alpha_k}{\beta_k}\}$$

which is given by:

$$p^{mon}_k(X) = \frac{1}{2}(C_k(X) + \frac{\alpha_k}{\beta_k})$$

Setting the monopoly price, the maximum profit at each market k is:

$$\frac{1}{4\beta_k}(\alpha_k - \beta_k C_k(X))^2$$

Then the *JPM* problem is as follows:

$$Maximize \quad \{\Pi(X) = \sum_{k=1}^{m} \frac{1}{4\beta_k}(\alpha_k - \beta_k C_k(X))^2 \ :|\ X^i\ |= f^i, X^i \subset L\}$$

4.1 Node Optimality Property

If the location candidates are the nodes and the points on the edges of a network, the following property holds.

Property 1 *There exists a set of nodes which is an optimal solution of (JPM) if for all $i \in F$ the marginal delivered cost, $dc^i(x,k)$, is a concave function for any market k when x varies along any edge of the network.*

Proof Let X be any set of location candidates. The minimum delivered cost to any market k is $C_k(X) = \min\{dc^i(y,k)\ :\ y \in X, i = 1, ..., r\}$. For $x \in X$, $C_k(X)$ is a concave function at x when x varies along any edge of the network assuming fixed the locations in X other than x. In such a case, $(\alpha_k - \beta_k C_k(X))^2$ is a convex function at x and the objective function $\Pi(X)$ is a sum of convex functions. Therefore $\Pi(X)$ is convex at x when x varies along any edge of the network, assuming fixed the locations in X other than x. This means that the maximum value of $\Pi(X)$ in that edge is reached at one of the nodes that define that edge. Let v such a node, if x is replaced by v in the set X, a new set X' is obtained for which $\Pi(X) \leq \Pi(X')$. Consequently, a *JPM* solution can be found in the set of nodes of the network. □

Therefore, if the location space is a network with nodes and points on the edges as location candidates, (*JPM*) can be solved taking L as the set of nodes if Property 1 holds. Concavity of marginal production cost and marginal transportation cost, and therefore concavity of delivered cost, is realistic in certain situations as it has been remarkable by many authors (see for instance Labbe and Hakimi 1991; Sarkar et al. 1997).

4.2 MILP Formulation for Different Production Costs

The set of facility locations $X = (X_1, X_2, \ldots, X_r)$ is represented by a vector $x = (x^1, x^2, \ldots, x^r)$ where $x^i = (x^i_1, x^i_2, \ldots, x^i_n)$ with components:

$$x^i_j = \begin{cases} 1 & \text{if location} \quad j \in X^i \\ 0 & otherwise. \end{cases}$$

For simplicity, let C^i_{kj} denote the delivered cost of firm i from location j to market k. The minimum delivered cost $C_k(X)$ can be determined through the following variables:

$$y^i_{kj} = \begin{cases} 1 & \text{if} \quad C^i_{kj} = C_k(X) \\ 0 & otherwise. \end{cases}$$

Let $D_k = \max\{C^i_{kj} : j \in L, i = 1, ..., r\}$ for all $k \in M$. With the previous variables the joint profit maximization problem becomes into the following Binary Integer Linear Programming problem:

$$Maximize \quad \sum_{k=1}^{m} \frac{1}{4\beta_k}(\alpha_k^2 - 2\alpha_k\beta_k \sum_{i=1}^{r} \sum_{j=1}^{n} C^i_{kj}y^i_{kj} + \beta_k^2 \sum_{i=1}^{r} \sum_{j=1}^{n} (C^i_{kj})^2 y^i_{kj})$$

$$\text{s.a.} \qquad \sum_{j=1}^{n} x^i_j = f^i \quad ; \quad \forall i \tag{5}$$

$$\sum_{i=1}^{r} x^i_j \leq 1 \quad ; \quad \forall j \tag{6}$$

$$y^i_{kj} \leq x^i_j \quad ; \quad \forall i, \quad \forall j, \quad \forall k \tag{7}$$

$$\sum_{i=1}^{r} \sum_{j=1}^{n} y^i_{kj} = 1 \quad ; \quad \forall k \tag{8}$$

$$\sum_{i=1}^{r} C^i_{kj}y^i_{kj} \leq \sum_{i=1}^{r} C^i_{kl}x^i_l + D_k(1 - \sum_{i=1}^{r} x^i_l) \quad ; \quad \forall l \neq j, \quad \forall k \tag{9}$$

$$x^i_j, y^i_{kj} \in \{0, 1\}; \quad \forall i, \quad \forall j, \quad \forall k \tag{10}$$

The objective function gives the joint profit of the firms. Constraint (5) indicates the number of facilities to be located by each firm. Constraints (6) mean that at most one facility is located at each location candidate. Constraints (7) guarantee that any market k can not be served from a firm i at location j if no facility of firm i is located at j. Constraints (8) means each market k will be served by only one firm from one facility. Constraints (9) determine the firm and the facility location at which the minimum delivered cost is obtained. Constraints (10) require that variables x^i_j and y^i_{kj} are binary.

An equivalent problem is obtained if constraints $y^i_{kj} \in \{0, 1\}$ in (10) are replaced by constraints $y^i_{kj} \geq 0$. Then the previous problem can be formulated as a *MILP* problem.

4.3 MILP Formulation for Equal Production Costs

If delivered costs are the same for all firms, the joint profit is determined by the location of the facilities, but it does not depend on the firm that owns each facility. Note that once f facility locations are fixed, the same joint profit is obtained for any assignment of f^i of those facilities to each firm i. Then the multi-facility joint profit maximization problem with r firms, each locating f^i facilities, is equivalent to a single facility joint profit maximization problem with one firm locating f facilities, where $f = f^1 + f^2 + ... + f^r$. Then the problem can be formulated in a simpler way as follows.

The set of facility locations $X = (X_1, X_2, \dots, X_r)$ can be represented by a vector $x = (x_1, x_2, \dots, x_n)$ with components:

$$x_j = \begin{cases} 1 & \text{if node } j \in X \\ 0 & \textit{otherwise.} \end{cases}$$

Let C_{kj} denote the delivered cost from location j to market k, then $C^i_{kj} = C_{kj}$, for $i = 1, \ldots, r$. The minimum delivered costs $C_k(X)$ can be determined through the following variables:

$$y_{kj} = \begin{cases} 1 & \text{if} \quad C_{kj} = C_k(X) \\ 0 & \textit{otherwise.} \end{cases}$$

Let $D_k = \max\{C_{kj} : j \in L\}$ for all $k \in M$. With the previous variables the joint profit maximization problem becomes the following Binary Integer Linear Programming problem:

$$\textit{Maximize} \quad \sum\nolimits_{k=1}^{m} \frac{1}{4\beta_k}(\alpha_k^2 - 2\alpha_k\beta_k \sum\nolimits_{j=1}^{n} C_{kj}y_{kj} + \beta_k^2 \sum\nolimits_{j=1}^{n} (C_{kj})^2 y_{kj})$$

$$\text{s.a.} \qquad \sum_{j=1}^{n} x_j = f \quad ; \tag{11}$$

$$y_{kj} \le x_j \quad ; \quad \forall j, \quad \forall k \tag{12}$$

$$\sum_{j=1}^{n} y_{kj} = 1 \quad ; \quad \forall k \tag{13}$$

$$C_{kj}y_{kj} \le C_{kl}y_l + D_k(1 - x_l) \quad ; \quad \forall l \ne j, \quad \forall k \tag{14}$$

$$x_j, y_{kj} \in \{0, 1\}; \quad \forall i, \quad \forall j, \quad \forall k \tag{15}$$

The objective function gives the joint profit of the firms. Constraint (11) indicates the total number of facilities to be located. Constraints (12) guarantee that any market can not be served from location j if no facility is located at j. Constraints (13) mean that for each market k only one facility is selected as the one with the minimum delivered cost to a market k. Constraints (14) guarantee that $y_{kj} = 1$ if j is the location with the minimum delivered cost to a market k. Constraints (15) require that variables x_j and y_{kj} are binary.

An equivalent problem is obtained if constraints $y_{kj} \in \{0, 1\}$ in (15) are replaced by constraints $y_{kj} \ge 0$. Then the previous problem can be formulated as a *MILP* problem.

5 An Empirical Study

We have studied the performance of the *MILP* formulations with some real data where both markets and location candidates are Spanish municipalities. Markets are municipalities with a population over 5000 inhabitants (1049 cities) which have been numbered from 1 to 1049 in decreasing population size, thus $M = \{1, 2, ..., 1049\}$. We have taken two sets of municipalities as location candidates, which are the municipalities with a population over 200000 inhabitants ($L = \{1, 2, ..., 24\}$) and 100000 inhabitants ($L = \{1, 2, ..., 54\}$), respectively (see Fig. 2). The population size and geographical coordinates of the Spanish municipalities can be seen on the website: http://www.um.es/geloca/gio/datos-espana-2015.txt.

Demand at each market k is given by a linear function:

$$q_k(p) = \begin{cases} \alpha_k - \beta_k p \ \ if \ \ 0 \le p \le \frac{\alpha_k}{\beta_k} \\ 0 \qquad \qquad otherwise \end{cases}$$

where α_k represents the maximum demand at municipality k and $\frac{\alpha_k}{\beta_k}$ is the maximum price the customers in market k are willing to pay for the product. Assuming that a maximum of one out of 1000 inhabitants buy the product, and that the maximum price the customers in market k are willing to pay is 1200, the following values of parameters α_k and β_k have been taken:

$$\alpha_k = \frac{\text{size of municipality } k}{1000}$$
$$\beta_k = \frac{\alpha_k}{1200}$$

We have solved 40 *JPM* test problems for firms with different production costs and 40 *JPM* test problems for firms with equal production costs, taking different combinations of the number of location candidates, the number of firms and the number of facilities of each firm which has been randomly generated in the interval [1,8]. After obtaining the joint profit maximization locations, the best response procedure has been applied to each test problem to generate a *NE* solution by taking the optimal solution of the corresponding *JPM* problem as starting solution of the iterative procedure. All MILP problems

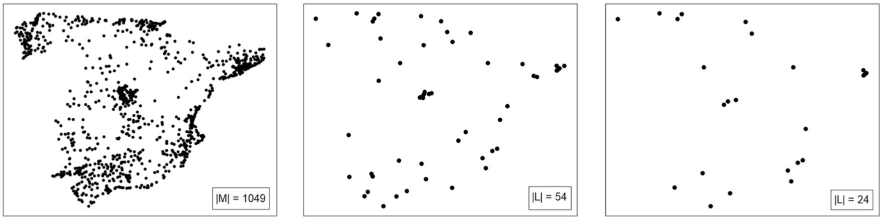

Fig. 2 Demand points and location candidates

involved have been solved with the software FICO Xpress Mosel (2014), 64 bits v.3.10.0 for Linux, on a computer with a processor Intel Core i7-6700 3.40 Ghzx8, RAM 8GB and OS Linux Ubuntu 64 bits.

5.1 Results for Firms with Different Production Costs

For simplicity we consider that marginal delivered costs are given by $dc^i(j,k) = pc^i + \mu d(j,k)$, where $d(j, k)$ is the distance between municipalities j and k; μ is the transportation cost per unit of distance; and pc^i is the production cost of firm i at any municipality j (for each firm it is assumed the same production cost at different location candidates). Distances $d(j, k)$ between any pair of municipalities j and k have been approximated by using the Harversine formula, which measure the distance between two geographical points from their longitudes and latitudes (see Mwemezi and Huang 2011). The transportation cost per unit of distance and the production cost have been randomly generated for each test problem in the intervals [0.1,0.3] and [50,70], respectively.

The results are shown in the Tables 3 and 4 in the Appendix. In each table: Column 1 shows the cardinality of the set of location candidates. Columns 2 and 3 give the number of facilities to be located by each firm, and their production costs, respectively. Column 4 shows the transportation cost per unit of distance. Columns 5 and 6 give the running time in minutes to generate the *JPM* and the *NE* solutions, respectively. Column 7 shows show the number of iterations of the best response procedure. Columns 8 and 9 show the joint profits obtained the *JPM* and the *NE* solutions, respectively. Finally, column 10 gives the percentage of joint profit decrease if the *JPM* solution is replaced by the *NE* solution.

A *NE* is found in a very small running time in comparison with the one to find a *JPM* solution. The number 1 in column *iter* implies that the *JPM* solution is also a *NE* solution. This occurs in all test problems but in three of them where the number of iterations to find a *NE* is 2 (the corresponding *JPM* solutions are not *NE*). Note that the joint profit obtained by a *JPM* solution is quite higher than the one obtained by the corresponding *NE* solution. Therefore, if the firms cooperate to set the monopoly price in each market, they would get a joint profit much higher than the joint profit prescribed by the *NE* that would result if the firms do not cooperate and set the equilibrium price in each market.

The results are summarized in Table 1. Columns 1 and 2 show the number of firms and the number of location candidates, respectively. Columns 3 and 4 give the average running time in minutes to generate the *JPM* and the *NE* solutions, respectively. Column 5 and 6 show the average joint profits obtained by the *JPM* and the

Table 1 Average results for different production costs

r	$\mid L \mid$	$\bar{t}_{JPM}$	$\bar{t}_{NE}$	$\overline{\Pi}_{JPM}$	$\overline{\Pi}_{NE}$	$\bar{p}(\%)$
2	24	1.15	0.01	9032.5	1737.2	80.67
	54	12.81	0.01	9008.1	1417.5	84.2
3	24	1.82	0	9072	1335.8	85.24
	54	18.81	0.01	9083.2	1348.5	85.11

Table 2 Average results for equal production costs

r	$\vert L \vert$	$\bar{t}_{JPM}$	$\bar{t}_{NE}$	$\bar{\Pi}_{JPM}$	$\bar{\Pi}_{NE}$	$\bar{p}(\%)$
2	24	0.11	0	9977.6	1150	88.28
	54	0.98	0.02	10018	1392.6	86.06
3	24	0.1	0	10055.3	957.4	90.46
	54	0.85	0.01	10062	1115.1	88.9

NE solutions, respectively. Finally, column 7 gives the average percentage of joint profit decrease if the *JPM* solution is replaced by the *NE* solution. Average running time to find a *JPM* solution increases approximately ten times when the number of location candidates increases from 24 to 54. The decrease in average joint profit if the *JPM* solution is replaced by the *NE* (firms do not cooperate and monopoly prices are replaced by equilibrium prices) is more than 80 % in all cases.

5.2 Results for Firms with Equal Production Costs

We now consider that $pc^i = 0$ and $dc^i(j,k) = \mu d(j,k)$, for each firm i. The total number of facilities f has been randomly generated in the interval [3,24] for each *JPM* test problem. With the aim of obtaining the *NE* solution from the corresponding *JPM* solution, the number of facilities f^i of each firm i has been randomly generated for each value of f, so that $f = \sum_{i=1}^{r} f^i$. The results are shown in Tables 5 and 6 where columns have the same meaning as in Tables 3 and 4, respectively.

Observe that running time to find a *JPM* solution for equal production costs is much smaller than the one for different production costs. For 54 location candidates the running time is around 1 min for equal production costs while it is around 18 min for different production costs. As it already happened when the production costs were different, a *NE* is found in a very small running time in comparison with the one to find a *JPM* solution. The *JPM* solution is also a *NE* in almost all test problems, only in three test problems for two firms is obtained that the *JPM* solution is not *NE* (the number of iterations to find a *NE* is 2). The joint profit obtained by a *JPM* solution is also higher than the one obtained by the corresponding *NE* solution.

The results are summarized in Table 2 where columns have the same meaning as in Table 1. Average running time to find a *JPM* solution increases approximately nine times when the number of location candidates increases from 24 to 54. The decrease in average percentage of joint profit if the *JPM* solution is replaced by the *NE* (firms do not cooperate and monopoly prices are replaced by equilibrium prices) is more than 86 % in all cases.

6 Conclusions and Future Research

The location-price decision problem for firms that sell an homogeneous product in spatially separated markets under a delivered pricing policy is studied. If the firms cooperate with the aim of joint profit maximization, they will set monopoly prices which are determined by facility locations. Then the problem is reduced to a location decision problem which is hard to solve.

For demand linear in price, it is shown that optimal locations can be obtained at the nodes of the transportation network that links location candidates and demand points, provided that the concavity assumption of the node optimality property holds. In such a case, the location decision problem can be linearized and two *MILP* models are developed to find joint profit maximization locations for different and equal production costs, respectively.

An empirical study is performed to test the proposed models and to compare the joint profit generated by the cooperative solution to the location decision problem with the one obtained by the Nash equilibrium solution of the location game that would result if the firms do not cooperate. Contrary to what occurs for demand fixed, where joint profit maximization locations are rarely *NE* (see Pelegrín et al. 2024), in almost all test problems is obtained that joint profit maximization locations are also *NE* of the corresponding location game. In all cases, the maximum joint profit got if firms set the monopoly price is much greater than the one got if they set the equilibrium price. Then cooperation could make each firm earn a profit higher than the one got if the firms do not cooperate. How to split the joint profit among the firms requires negociation and it will be a subject for future research.

Appendix

Table 3 Results for two firms and different production costs

$\mid L \mid$	$[f^1,f^2]$	$[pc^1,pc^2]$	μ	t_{JPM}	t_{NE}	*iter*	Π_{JPM}	Π_{NE}	$p(\%)$
	[1,5]	[52,58]	0.14	1.12	0.01	1	9083	1126	87.6
	[4,1]	[52,67]	0.15	1.13	0.01	1	9069	2489	72.6
	[6,8]	[55,57]	0.28	1.16	0.00	1	9097	1287	85.9
	[7,8]	[51,69]	0.13	1.15	0.00	1	9215	916	90.1
24	[1,2]	[58,57]	0.18	1.15	0.02	2	8760	2076	76.3
	[8,6]	[70,58]	0.28	1.12	0.00	1	9002	1627	81.9
	[1,4]	[55,58]	0.1	1.18	0.01	1	9098	825	90.9
	[3,8]	[51,54]	0.21	1.13	0.01	1	9171	1455	84.1
	[1,5]	[60,58]	0.3	1.15	0.01	1	8852	4074	54.0
	[6,1]	[63,64]	0.16	1.16	0.00	1	8978	1497	83.3
	[8,6]	[60,69]	0.2	12.57	0.01	1	9055	1193	86.8
	[4,3]	[61,64]	0.21	13.03	0.01	1	8942	1636	81.7
	[3,5]	[60,56]	0.26	13.52	0.01	1	8997	1579	82.4
	[2,3]	[52,53]	0.13	13.11	0.01	1	9126	1141	87.5
54	[7,3]	[62,60]	0.22	12.94	0.01	1	9016	1491	83.5
	[2,3]	[56,55]	0.15	12.48	0.02	2	9049	1274	85.92
	[5,6]	[57,63]	0.17	12.74	0.01	1	9099	1004	89.0
	[8,5]	[58,65]	0.12	12.44	0.01	1	9144	764	91.6
	[3,1]	[64,62]	0.15	12.32	0.01	1	8842	1299	85.3
	[2,3]	[70,53]	0.27	12.93	0.02	1	8811	2794	68.3

Table 4 Results for three firms and different production costs

$\mid L \mid$	$[f^1,f^2,f^3]$	$[pc^1,pc^2,pc^3]$	μ	t_{JPM}	t_{NE}	*iter*	Π_{JPM}	Π_{NE}	$p(\%)$
	[1,8,4]	[67,52,62]	0.22	1.73	0.00	1	9160	1263	86.2
	[5,4,8]	[53,69,68]	0.3	1.70	0.00	1	9043	1352	85.0
	[8,1,3]	[50,64,51]	0.12	1.87	0.00	1	9279	709	92.4
	[6,1,1]	[53,58,63]	0.23	2.03	0.01	1	9072	1848	79.6
24	[2,3,1]	[62,60,54]	0.17	1.80	0.01	1	9005	1401	84.4
	[1,5,4]	[64,59,70]	0.2	1.80	0.00	1	9021	1645	81.8
	[4,1,3]	[62,70,55]	0.27	1.91	0.01	1	8959	1969	78.0
	[1,8,6]	[70,58,53]	0.24	1.73	0.00	1	9140	1036	88.7
	[6,7,4]	[66,65,62]	0.23	1.72	0.00	1	9011	865	90.4
	[1,6,7]	[59,68,60]	0.24	1.88	0.01	1	9030	1270	85.9
	[4,6,6]	[54,59,58]	0.1	18.54	0.01	1	9216	514	94.4
	[4,4,2]	[60,62,54]	0.1	18.49	0.01	1	9161	810	91.2
	[2,7,1]	[70,55,65]	0.22	18.53	0.01	1	9085	2216	75.6
	[2,5,7]	[64,61,62]	0.29	20.77	0.01	1	8999	1659	81.6
54	[3,4,2]	[50,60,60]	0.13	18.21	0.02	2	9127	810	91.13
	[4,4,8]	[62,63,53]	0.2	18.62	0.01	1	9172	1141	87.6
	[2,4,1]	[53,59,67]	0.24	18.76	0.01	1	8984	1739	80.6
	[6,1,2]	[63,68,52]	0.25	18.28	0.01	1	9006	1871	79.2
	[4,1,3]	[55,51,59]	0.2	19.09	0.01	1	9087	1178	87.0
	[2,3,3]	[57,59,65]	0.22	18.77	0.01	1	8995	1547	82.8

Table 5 Results for two firms and equal production costs

$\mid L \mid$	$[f^1,f^2]$	μ	t_{JPM}	t_{NE}	*iter*	Π_{JPM}	Π_{NE}	$p(\%)$
	[7,6]	0.16	0.10	0.00	1	10091	933	90.8
	[5,7]	0.11	0.10	0.00	1	10123	533	94.7
	[2,7]	0.10	0.12	0.00	1	10111	984	90.3
	[6,4]	0.11	0.11	0.01	1	10112	887	91.2
24	[5,3]	0.23	0.12	0.01	1	9964	1387	86.1
	[1,1]	0.29	0.14	0.01	1	9110	2786	69.4
	[7,7]	0.11	0.09	0.00	1	10130	579	94.3
	[8,4]	0.13	0.10	0.00	1	10108	709	93.0
	[5,2]	0.14	0.13	0.01	2	10037	1527	84.8
	[4,1]	0.13	0.13	0.00	1	9990	1175	88.2
	[5,7]	0.15	0.91	0.01	1	10093	894	91.1
	[7,1]	0.16	0.99	0.03	1	10037	2348	76.6
	[1,5]	0.16	1.04	0.02	1	9988	2495	75.0
	[3,8]	0.12	0.98	0.02	2	10109	840	91.7
54	[1,7]	0.11	0.99	0.02	1	10090	1504	85.1
	[2,5]	0.15	1.01	0.01	1	10026	1409	85.9
	[4,2]	0.13	0.98	0.01	1	10029	1133	88.7
	[2,4]	0.11	0.97	0.01	1	10056	994	90.1
	[1,1]	0.15	1.12	0.05	2	9628	1569	83.7
	[6,8]	0.12	0.83	0.01	1	10124	740	92.7

Table 6 Results for three firms and equal production costs

$\|L\|$	$[f^1,f^2,f^3]$	μ	t_{JPM}	t_{NE}	*iter*	Π_{JPM}	Π_{NE}	$p(\%)$
	[4,1,1]	0.11	0.13	0.01	1	10055	994	90.1
	[2,7,1]	0.17	0.11	0.00	1	10060	1224	87.8
	[4,6,4]	0.1	0.09	0.00	1	10137	454	95.5
	[5,6,7]	0.22	0.09	0.00	1	10065	384	96.2
24	[6,7,2]	0.2	0.09	0.00	1	10071	967	90.4
	[6,2,4]	0.28	0.10	0.01	1	9995	1773	82.3
	[7,5,7]	0.25	0.09	0.00	1	10047	541	94.6
	[4,5,2]	0.28	0.12	0.01	1	9981	1816	81.8
	[7,6,4]	0.25	0.09	0.00	1	10043	800	92
	[7,7,6]	0.17	0.09	0.00	1	10099	621	93.9
	[1,8,6]	0.14	0.84	0.01	1	10115	857	91.5
	[6,7,3]	0.26	0.78	0.01	1	10045	1179	88.3
	[7,7,3]	0.22	0.80	0.01	1	10075	1091	89.2
	[8,7,3]	0.23	0.79	0.01	1	10075	1054	89.5
54	[4,7,7]	0.19	0.80	0.01	1	10097	836	91.7
	[2,5,5]	0.28	0.86	0.01	1	9995	1749	82.5
	[3,1,7]	0.16	0.85	0.01	1	10077	1039	89.7
	[6,2,2]	0.11	0.99	0.01	1	10112	593	94.1
	[5,5,1]	0.27	0.99	0.01	1	9989	1427	85.7
	[6,7,4]	0.28	0.81	0.01	1	10040	1326	86.8

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Declarations

Competing Interests The authors declare no competing interests.

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