TESIS DOCTORAL

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UNIVERSIDAD CATÓLICA
DE MURCIA

ESCUELA INTERNACIONAL DE DOCTORADO

Programa de Doctorado en Tecnologías de la Computación e Ingeniería Ambiental

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Diseño, implementación y análisis dinámico de familias paramétricas de métodos iterativos, para resolver ecuaciones y sistemas no lineales

Autor:
D. José Javier Padilla Abellán

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TESIS POR COMPENDIO

Esta tesis doctoral es un compendio de publicaciones científicas publicadas en revistas indexadas en el JCR. Los artículos científicos, así como sus correspondientes referencias, están listados a continuación:

- Padilla, J.J.; Chicharro, F.I.; Cordero, A.; Torregrosa, J.R. Parametric Family of Root-Finding Iterative Methods: Fractals of the Basins of Attraction. Fractal Fract. 2022, 6, 572. https://doi.org/10.3390/fractalfract6100572
- Padilla, J.J.; Chicharro, F.I.; Cordero, A.; Hernández-Díaz, A.M.; Torregrosa, J.R. Parametric Iterative Method for Addressing an Embedded-Steel Constitutive Model with Multiple Roots. Mathematics 2023, 11, 3275. https://doi.org/10.3390/math11153275
- Padilla, J.J.; Chicharro, F.I.; Cordero, A.; Hernández-Díaz, A.M.; Torregrosa, J.R. A Class of Efficient Sixth-Order Iterative Methods for Solving the Nonlinear Shear Model of a Reinforced Concrete Beam. Mathematics 2024, 12, 499. https://doi.org/10.3390/math12030499

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RESUMEN

En las últimas décadas, la producción científica sobre los métodos numéricos de varios pasos para resolver ecuaciones no lineales o sistemas de ecuaciones no lineales se ha incrementado. Este hecho viene motivado porque los métodos numéricos de un único paso requieren en sus expresiones derivadas de órdenes superiores para alcanzar mayor velocidad de convergencia, aunque la velocidad con la que converge un método es solo una de las características que ayudan a catalogar los métodos numéricos. El objetivo de esta tesis es diseñar nuevos esquemas iterativos de punto fijo, multipunto y sin memoria, los cuales superen las propiedades de los métodos ya conocidos, como en la eficiencia computacional y la estabilidad. En esta tesis, presentamos dos familias de métodos iterativos que dependen de parámetros, y un método para aproximar las soluciones de sistemas no lineales, que han sido diseñados mediante dos técnicas, la de composición y la de funciones peso. Se desarrolla un estudio basado en la dinámica discreta compleja, que ayuda a elegir aquellos miembros de la familia más estables. Se realizan pruebas numéricas que muestran cómo estos miembros son útiles para la aproximación de soluciones con funciones no polinómicas, como la famosa ecuación de Colebrook-White, usada en ingeniería hidráulica, y la ecuación constitutiva del acero embebido en hormigón armado. En este último caso, se trata de un modelo de comportamiento del acero con endurecimiento (hardening) debido al efecto sobre la armadura de acero de la tensorrigidez del hormigón envolvente, y que además presenta una región algebraica de solubilidad, lo que hace aún más interesante, si cabe, su resolución mediante métodos iterativos de orden superior.

PALABRAS CLAVE

Ecuaciones y sistemas no lineales, análisis numérico; métodos iterativos; ingeniería civil; diseño de sistemas de cálculo, tecnología del hormigón.

ABSTRACT

In the last decades the scientific production on multistep numerical methods for solving nonlinear equations or systems of nonlinear equations, has increased, this fact is motivated because the single-step numerical methods require in their derivative expressions of higher orders to achieve higher convergence speed, although the speed with which a method converges is only one of the characteristics that help to catalog the numerical methods. The goal of this thesis is to design new fixed-point, multipoint, memoryless iterative schemes, which outperform the properties of the already known methods, such as in computational efficiency and stability. In this thesis, we present two families of parameter-dependent iterative methods, and a method for approximating the solutions of nonlinear systems, which have been designed using two techniques, composition and weight functions. A study based on complex discrete dynamics is developed, which helps to choose those members of the most stable family. Numerical tests are performed showing how these members are useful for the approximation of solutions with non-polynomial functions, such as the famous Colebrook-White equation, used in hydraulic engineering, and the constitutive equation of steel embedded in reinforced concrete. In the latter case, it is a model of steel behavior with hardening due to the effect on the steel reinforcement of the tensile rigidity of the enveloping concrete, and which also presents an algebraic region of solubility, which makes it even more interesting, if possible, to solve it by means of higher order iterative methods.

KEYWORDS

Nonlinear equations and systems, numerical analysis; iterative methods; civil engineering; design of calculation systems, concrete technology.

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A Dios.

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De está vida sacarás
lo que metas nada más
y un poquito menos.
Pedro Padilla García (1945-2016)

Índice general

Capítulo 1

Introducción

Aproximar las soluciones de ecuaciones y sistemas no lineales, también llamadas ceros o raíces, es uno de los problemas más comunes en las matemáticas aplicadas, desde la antigüedad hasta la actualidad. El problema de encontrar una aproximación, para la raíz de una ecuación, se puede rastrear por los menos hasta el año 1700 a.C., en el que una tabla cuneiforme en la Colección Babilónica de Yale [1], provee un número sexagesimal (base 60) equivalente a 1.414222 como una aproximación para $\sqrt{2}$, un resultado que es preciso dentro de 10^{-5} [2]. Herón de Alejandría, desarrolló una fórmula que calculaba raíces cuadradas en el siglo I d.C., en la cual usa una aproximación a la solución, para conseguir mejores aproximaciones. Hasta el siglo XVI sólo se podían encontrar soluciones analíticas a polinomios de segundo grado, entonces el italiano Gerolamo Cardano publicó un método para resolver ecuaciones cúbicas, originalmente desarrollado por Scipione y Tartaglia. Ludovico Ferrari resolvería las ecuaciones de cuarto grado, ese mismo siglo. Numerosos matemáticos intentaron resolver ecuaciones de orden superior, pero en el siglo XIX se demostró que no es posible sin el uso de técnicas numéricas, obteniendo aproximaciones a dichas raíces. Con la revolución informática, que ha desarrollado rápidamente las computadoras digitales, los métodos numéricos han ganado importancia, ya que han podido aplicarse para resolver problemas que se presentan en numerosos campos del conocimiento humano, como el cálculo de trayectoria de satélites [3], crecimiento de células tumorales [4], reacciones químicas [5] o transferencia de calor [6], son solo algunos ejemplos. Se han diseñado nuevos métodos, controlando diferentes características, como la eficiencia computacional (tiempo de ejecución del método), los errores en las aproximaciones, o la dependencia de la estimación inicial para asegurar la convergencia.

La computación científica se basa en la iteración, donde se repite un proceso, con el que se obtiene una sucesión de valores, hasta que se obtiene el resultado deseado.

En esta tesis abordamos la solución de este tipo de problemas mediante esquemas iterativos de punto fijo. Un punto fijo de una función $\Phi(x)$ es un número real ξ tal que $\xi = \Phi(\xi)$. Gráficamente es la intersección de la curva $\Phi(x)$ con la recta $y = x$. Esta función $\Phi(x)$ es con la que calcularemos los sucesivos términos, a partir de un valor inicial, x_0, produciendo la sucesión de valores x_k, que resulta del proceso iterativo

$$x_{k+1} = \Phi(x_k).$$

La sucesión nos daría el siguiente patrón

$$\{x_0, x_1 = \Phi(x_0), x_2 = \Phi(x_1), \ldots, x_k = \Phi(x_{k-1}), x_{k+1} = \Phi(x_k) \ldots\}.$$

Si esta sucesión tiende a un límite, estará acotada, y en tal caso diremos que converge. Si no converge, la sucesión puede diverger, ser periódica o caótica.

Para resolver ecuaciones no lineales $f(x) = 0$, el esquema más utilizado es el método de Newton, también llamado, de Newton-Raphson, es de un paso, sin memoria y sigue la siguiente expresión:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, ... \tag{1.1}$$

siendo $f : I \subseteq \mathbb{R} \to \mathbb{R}$, y siempre que $f'(x) \neq 0$ en el intervalo en el que se encuentra la raíz α, geométricamente este método, comienza a iterar desde una estimación inicial x_0, próxima a la raíz α, y a continuación se extiende una recta tangente desde el punto $(x_0, f(x_0))$, el punto donde la recta tangente corta al eje x suele representar la estimación mejorada x_1, volviendo a repetir el proceso desde x_1. El método de Newton converge cuadráticamente (el número de cifras significativas se duplica en cada iteración), para raíces simples.

Al contener una derivada, es posible aplicar este método a sistemas de ecuaciones no lineales, simplemente sustituyendo la derivada de la función por la matriz Jacobiana [7].

Los métodos de un paso, como el de Newton, Halley, Laguerre, Euler-Cauchy, para alcanzar orden p, deben contener en su esquema iterativo, derivadas de al menos orden $p - 1$ [8]. Estos métodos son simples y eficaces, pero convergen lentamente, y su aplicación a problemas reales está restringida. Superar la convergencia cuadrática de esquemas iterativos, no es posbile con métodos de un paso y utilizando solo la primera derivada.

Esta es la razón por la que los métodos multipunto centraron la atención de los matemáticos, en las últimas décadas del siglo XX. En [9] se recogen muchos de estos métodos multipunto. Aumentar el orden de convergencia, añadiendo pasos a los métodos, mediande la técnica de composición [10] , suele incrementar el número de evaluaciones funcionales en cada paso, la relación entre el orden de convergencia y el número de evaluaciones funcionales, permite medir la eficiencia de un esquema iterativo. Dicha relación recibe el nombre de índice de eficiencia, y varios autores la han definido de diferentes formas ([8],[11]). La presentó en 1960, el matemático ucraniano Alexander Ostrowski, quien desarrolló el primer método de dos pasos, de cuarto orden de convergencia [7], que al necesitar tres evaluaciones funcionales, es óptimo, concepto que veremos con detalle en el Capítulo 2, entre otros. En 1964, el matemático alemán, Joseph Frederick Traub, publicó un libro [8] en el que estudiaba ampliamente los métodos multipunto, de orden alto, los cuales obtienen gran precisión en las aproximaciones. En los últimas décadas, el interés por diseñar métodos multipaso con mayor eficiencia, se ha incrementado.
En esta tesis presentamos tres métodos multipasos, dos para aproximar las soluciones de una ecuación no lineal y uno para resolver sistemas de ecuaciones no lineales, haciendo uso de la técnica de composición y de funciones peso. Hemos estudiado la convergencia de los mismos, su eficiencia, y los hemos comparado con métodos numéricos ya conocidos, aplicándolos sobre problemas académicos y también sobre problemas aplicados. Con ayuda de la dinámica compleja, hemos podido seleccionar los miembros de las familias de métodos iterativos diseñadas, aquellos con mayor estabilidad. Las pruebas numéricas se han realizado sobre problemas teóricos, y problemas aplicados a la ingeniería estructural.

En el Capítulo 2 de esta tesis se detallan los conceptos básicos en los que se apoyan los artículos. Proporcionaremos las definiciones básicas de métodos numéricos para resolver ecuaciones y sistemas de ecuaciones no lineales, la definición de método óptimo, las técnicas usadas para el diseño de estos métodos, conceptos de dinámica compleja y cómo utilizarlos para seleccionar los miembros de las familias más eficientes.

La primera aportación es una familia de métodos, uniparamétrica, de cuarto orden de convergencia, los cuales son esquemas óptimos según la conjetura de Kung y Traub [12] y se analiza mediante herramientas de dinámica compleja los elementos más estables.

El segundo trabajo, es una mejora de la primera familia de métodos, para calcular funciones con raíces múltiples, en este caso los elementos de la clase de métodos iterativos son óptimos y se analiza la familia usando

herramientas de la dinámica compleja para seleccionar los elementos más estables.

Y por último, se presenta un método para resolver sistemas de ecuaciones, de tres pasos, con orden de convergencia seis, dicho esquema iterativo es eficiente, desde el punto de vista del coste computacional y robusto, a juzgar por las pruebas numéricas realizadas. Esta tesis finaliza describiendo futuras líneas de trabajo, y problemas abiertos, que nos hemos encontrado al desarrollar los métodos que se presentan. Pretendemos abordar dichos problemas en el futuro.

Capítulo 2

Conceptos básicos

2.1. Métodos iterativos escalares

Los métodos analíticos resuelven escasas ecuaciones y sistemas que se plantean en problemas de simulación de transporte de fluidos en tuberías [13], equilibrio químico en reactores [14], modelos estructurales [15], robótica [16], meteorología [17]. Por este motivo cobran interés los métodos numéricos. En esta tesis se describen esquemas iterativos de punto fijo con la siguiente expresión,

$$x_{k+1} = \Phi(x_k), \qquad k = 0, 1, 2..., \qquad \Phi : \mathbb{R} \to \mathbb{R}$$

donde $\Phi : \mathbb{R} \to \mathbb{R}$ y x_k es una aproximación a la raíz α, aislada en un intervalo $[a, b]$. El método comienza con una estimación inicial $x_0 \in [a, b]$, suficientemente cercana a la raíz α. A Φ la llamamos función de iteración o esquema iterativo.
En las últimas décadas se han diseñado numerosos métodos iterativos [18, 19, 20, 21] para resolver ecuaciones no lineales $f(x) = 0$, donde $f : I \subseteq \mathbb{R} \to \mathbb{R}$ siendo I un intervalo abierto. La mayoría de estos nuevos métodos, son modificaciones del método de Newton. A continuación describiremos diferentes tipologías dentro del conjunto de procesos iterativos.

2.1.1. Clasificación de métodos numéricos de punto fijo

Numerosas publicaciones presentan análisis de los métodos de Newton, Halley o Chebyshev [22, 23]. Los métodos iterativos podrían clasificarse atendiendo a alguna de sus características, por ejemplo, por los pasos que

lo componen:

- De un paso: la solución exacta es aproximada con x_{k+1}, desde una aproximación anterior x_k, como el de Newton (1.1)

$$x_{k+1} = \Phi(x_k), \qquad k = 0, 1, 2, ...$$

- Multipaso (predictor-corrector): se usan varias evaluaciones funcionales de la función no lineal o sus derivadas en puntos intermedios entre x_k y la nueva aproximación x_{k+1}

$$\begin{aligned} y_k &= \Psi(x_k), \\ x_{k+1} &= \Phi(x_k, y_k), \qquad k = 0, 1, 2, ... \end{aligned}$$

Un ejemplo de método multipaso es el método de Simpson

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \frac{6f(x_k)}{f'(x_k) + 4f'\left(\dfrac{x_k + y_k}{2}\right) + f'(y_k)}, \ k = 0, 1, 2, \ldots \end{aligned} \tag{2.1}$$

o la familia King [24]

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(x_k) + \beta f(y_k)}{f(x_k) + (\beta - 2) f(y_k)} \frac{f(y_k)}{f'(x_k)}, \ k = 0, 1, 2, \ldots \end{aligned} \tag{2.2}$$

Si atendemos a la cantidad de iterados empleados para calcular la nueva estimación:

- Con memoria:

$$x_{k+1} = \Phi(x_k, x_{k-1}, ..., (x_{k-r}, x_{k-m+1}, \ldots, x_{k-1}, x_k), m \leq k, k \geq 0.$$

  Por ejemplo el método de la secante requiere dos iterados previos para obtener el siguiente valor de la iteración

$$x_{k+1} = x_k - \frac{f(x_k)}{f[x_k, x_{k-1}]} \qquad k = 0, 1, 2, \ldots$$

siendo la diferencia dividida entre dos puntos

$$f[x_{k-1}, x_k] = \frac{f(x_{k-1}) - f(x_k)}{x_{k-1} - x_k}.$$

- Sin memoria:

$$x_{k+1} = \Phi(x_k), \qquad k = 0, 1, 2, \ldots.$$

El método Super-Halley es un método sin memoria que utiliza únicamente el iterado anterior, para aproximar el siguiente

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \left[\frac{L_f(x_k) - 2}{2(L_f(x_k) - 1)} \right], \qquad k = 0, 1, 2, \ldots$$

$$L_f = \frac{f(x_k) f''(x_k)}{f'(x_k)^2}.$$

Por utilizar derivadas o no, en su expresión iterativa:

- Con derivadas de la función no lineal, de diferentes órdenes, como el matemático francés Laguerre (1878)

$$x_{k+1} = x_k - \frac{\lambda f(x_k)}{f'(x_k) \pm \sqrt{(\lambda - 1)^2 \left[f'(x_k)^2 - \frac{\lambda}{\lambda - 1} f(x_k) f''(x_k) \right]}},$$

donde λ es un parámetro distinto de 0 y 1. Podemos encontrar modificaciones de este método en [25].

O el método de Chebyshev, cuya expresión iterativa es

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \left[1 + \frac{L_f(x_k)}{2} \right], \qquad k = 0, 1, 2, \ldots \tag{2.3}$$

$$L_f = \frac{f(x_k) f''(x_k)}{f'(x_k)^2}.$$

- Libres de derivadas. Son muy útiles si la función f no es derivable o su derivada es difícil de obtener. En su lugar utilizan las diferencias divididas, o aproximaciones provenientes de la derivación numérica, central, progresiva o regresiva, un ejemplo es el método de dos pasos que se presenta en este artículo [26], o el clásico método de Steffensen de un paso, cuya expresión es

$$x_{k+1} = x_k - \frac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}, \quad k = 0, 1, 2, \ldots \quad (2.4)$$

  Si requieren un intervalo cerrado de valores iniciales, o una estimación inicial:

  - Cerrados, si requieren como estimación inicial, los extremos de un itervalo $[a, b]$. Convergen, mientras $f(a) \cdot f(b) < 0$, o lo que es lo mismo, que el intervalo contenga a la raiz. Los métodos de la bisección y regula falsi, son algunos ejemplos.
  - Abiertos, o de punto fijo, en estos métodos no es necesario que la raiz esté contenida en un intervalo, parten de una estimación inicial, o semilla, la convergencia no está asegurada si la estimación inicial está lejos de la solución buscada. Algunos métodos abiertos clásicos son el método de Newton o Steffensen (2.4).

  Los métodos iterativos nunca proveen una solución exacta, incluso en precisión infinita. Por tanto se tiene que establecer un criterio de parada.

  El error absoluto en el paso k de un proceso iterativo se denota por $e_k^a = |x_k - \alpha|$ y se puede aproximar mediante la diferencia con el iterado anterior, $e_k^a \approx |x_k - x_{k-1}|$. Así mismo, el error relativo se define como $\tilde{e_k} = \frac{|x_k - \alpha|}{|\alpha|}$, aproximándose por $\tilde{e_k} \approx \frac{|x_k - x_{k-1}|}{|x_k|}$.
  Para evaluar el comportamiento de cada uno de los esquemas o clases de métodos iterativos diseñados, se llevan a cabo diferentes experimentos numéricos. En ellos se compara el desempeño de dichos procesos iterativos sobre ecuaciones académicas y problemas aplicados, comparando los resultados obtenidos con los de otros esquemas existentes en la literatura. El criterio de parada utilizado en las pruebas numéricas que se realizan para cada método iterativo, es la diferencia de los dos últimos iterados, en valor absoluto, $|x_{k+1} - x_k|$, al que se le suma, el valor de la función en el último iterado, $|f(x_{k+1})|$, $|x_{k+1} - x_k| + |f(x_{k+1})|$. De este modo se comprueba, no sólo que el método converge, sino que lo hace a una solución de nuestro problema.

2.1.2. Orden de convergencia y eficiencia de los métodos

Todo esquema iterativo para resolver ecuaciones $f(x) = 0$ necesita conocer al menos una estimación inicial x_0, para encontrar la raíz α. Si la estimación inicial o semilla, no está próxima a la raiz, los métodos convergerán lentamente en las primeras iteraciones, y su eficiencia computacional se verá penalizada. La elección de una buena estimación inicial x_0 no es un proceso sencillo, sobre todo en los problemas multidimensionales.

Si la sucesión $\{x_k\}_{k\geq 0}$, generada por un método iterativo, se aproxima a la raiz α, diremos que converge

$$\lim_{k\to\infty} x_k = \alpha,$$

y tiene orden de convergencia local p, si encontramos constantes $C > 0$ y $p \in \mathbb{R}$ tales que

$$\lim_{k\to\infty} \frac{|x_{k+1} - \alpha|}{|x_k - \alpha|^p} = C.$$

Si se tiene que $p = 1$, entonces $0 < C < 1$ y podemos afirmar que el método tiene convergencia lineal.
Si $p > 1$ y $C > 0$, entonces la convergencia del método es de orden p (cuadrática si $p = 2$, cúbica si $p = 3$...). En las últimas décadas numerosos investigadores han publicado métodos iterativos de alto orden de convergencia [27, 28, 29, 30, 31]. En esta memoria presentaremos dos familias de esquemas iterativos con convergencia de orden cuatro, para estimar raíces simples y múltiples de ecuaciones no lineales y un método para estimar soluciones no singulares de sistemas de ecuaciones no lineales, de orden seis.

Denotemos por $e_k = x_k - \alpha$ al error cometido en la iteración k. Cualquier método de orden de convergencia p satisface la llamada ecuación del error cuya expresión es

$$e_{k+1} = Le_k^p + \mathcal{O}(e_k^{p+1}),$$

donde $L \in \mathbb{R}$.

En la práctica, es importante conocer algunas características de los esquemas iterativos, por ejemplo, el número de operaciones necesarias para conseguir la precisión deseada o el tiempo de ejecución del

programa. Obviamente, un método es más eficiente, cuando su coste computacional es menor. En otras palabras, un método más eficiente es el que satisface el criterio de parada indicado, con el menor tiempo de cálculo. Utilizamos dos índices relacionados con la eficiencia de cada método, para poder cotejarlos, estos son, el índice de eficiencia definido por Ostrowski en [11] $I = p^{1/d}$ y el índice de eficiencia computacional definido en [32] como $IC = p^{1/(d+op)}$, siendo p el orden de convergencia del esquema, d el número de evaluaciones funcionales por iteración, y op, el número de (productos/cocientes) por iteración. El segundo índice es más apropiado en la resolución de sistemas de ecuaciones.

Por otra parte, el orden de convergencia de un método puede aproximarse en la práctica de diferentes formas. Una aproximación directa basada en la definición de orden es la tasa de convergencia

$$\frac{|x_{k+1} - x_k|}{|x_k - x_{k-1}|^p} = cte, \qquad \forall k \geq k_0.$$

Weerakoon y Fernando definen el Orden de Convergencia Computacional (COC) en [33] como sigue,

$$p \approx COC = \frac{\ln\left(|x_{k+1} - \alpha|/|x_k - \alpha|\right)}{\ln\left(|x_k - \alpha|/|x_{k-1} - \alpha|\right)}. \tag{2.5}$$

Sin embargo el valor exacto de la raíz α es desconocido, en la práctica, y en esta memoria utilizaremos el Orden de Convergencia Computacional (ACOC) definido en [34] como

$$p \approx ACOC = \frac{\ln\left(|x_{k+1} - x_k|/|x_k - x_{k-1}|\right)}{\ln\left(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|\right)}. \tag{2.6}$$

La conjetura de Kung y Traub [35] establece que un método iterativo sin memoria, para encontrar una raíz simple, de una ecuación escalar, es óptimo, si el orden de convergencia del método es igual a 2^{d-1}, con $d \in \mathbb{N}$. Al requerir dos evaluaciones funcionales ($d = 2$), una del valor de la función y otra de su primera derivada, el método de Newton es un método óptimo de segundo orden de convergencia.

2.1.3. Ejemplos clásicos de métodos iterativos

Métodos para raíces simples

En este apartado vamos a revisar dos métodos de un punto, para resolver ecuaciones no lineales $f(x) = 0$, con una raíz simple α, en el intervalo $[a, b]$. Donde $f \in \mathbb{C}^1[a, b]$ y $f'(x) \neq 0$ para $x \in [a, b]$.

Los métodos de Halley (2.7) y Chebyshev (2.3) son de un paso y utilizan derivadas de órdenes superiores en sus expresiones iterativas

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \left[1 + \frac{L_f(x_k)}{2 - L_f(x_k)} \right], \qquad k = 0, 1, 2, \ldots \quad (2.7)$$

$$L_f = \frac{f(x_k) f''(x_k)}{f'(x_k)^2}.$$

El método de Newton es el más utilizado para comparar con nuevos métodos iterativos.

El método de Jarratt (2.8) [36], alcanza orden cuatro, es un método de dos pasos y sin memoria:

$$\begin{aligned} y_k &= x_k - \frac{2}{3} \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \frac{3f'(y_k) + f'(x_k)}{6f'(y_k) - 2f'(x_k)} \frac{f(x_k)}{f'(x_k)}, \ k = 0, 1, 2, \ldots \end{aligned} \quad (2.8)$$

Los métodos multipaso con derivadas, como el de Newton y Jarrat son directamente extensibles a sistemas de ecuaciones no lineales.

Métodos para raíces múltiples

Los métodos anteriores solo resuelven ecuaciones no lineales con raíces simples, pero muchos problemas presentan raíces múltiples. Además la convergencia de estos métodos se reduce a lineal, en el mejor de los casos. Por esta razón es necesario desarrollar nuevos esquemas iterativos que permitan la estimación de este tipo de raíces como se realiza en [37, 38, 39, 40, 41].

Sea $m \geq 1$ la multiplicidad de una raíz α de una función

$$f(x) = (x - \alpha)^m g(x),$$

donde g satisface $g(\alpha) \neq 0$. Uno de los métodos de un punto, usados frecuentemente en la aproximación de raíces múltiples, es el método de Newton modificado, también llamado método de Rall [42]

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, ... \tag{2.9}$$

El método de Rall requiere el conocimiento de la multiplicidad de las raices. En el caso de que desconozcamos m, podemos aplicar el método de Schröder [43]

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{[f'(x_k)]^2 - f(x_k)f''(x_k)}, \quad k = 0, 1, 2, ... \tag{2.10}$$

que puede obtenerse al aplicar el esquema de Newton sobre la función $\frac{f(x)}{f'(x)}$. Mientras que el llamado método de Rall es óptimo, el conocido esquema de Schröder no lo es, aunque como contrapartida no necestia el conocimiento previo de la multiplicidad buscada. Además, si deseamos obtener más de una raíz del mismo problema, debemos cambiar la multiplicidad con el método de Rall, mientras que con el de Schröder solo necesitaremos cambiar la estimación inicial.

2.1.4. Diseño de métodos iterativos

Los métodos de un punto, que utilizan p evaluaciones funcionales por iteración, tienen una limitación en su orden de convergencia de como máximo p.

Para incrementar el orden de convergencia, se puede utilizar la técnica de composición de métodos, obteniendo así métodos multipunto, como en [44, 45], que fue generalizada por Traub.

Sean $g_1(x)$, $g_2(x)$ dos funciones de punto fijo para $f(x) = 0$. Se consideran los esquemas iterativos $x_{k+1} = g_1(x_k)$ y $x_{k+1} = g_2(x_k)$ de órdenes p_1 y p_2, respectivamente. Así, el orden de convergencia del método iterativo asociado a la función de punto fijo es $p_1 \cdot p_2$ [8].

Un ejemplo de esta técnica, es el método Newton-Halley, cuyo esquema iterativo es

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(y_k)}{f'(y_k) - \dfrac{f(y_k)f''(y_k)}{2f'(y_k)}}, \ k = 0, 1, 2, \ldots \end{aligned}$$

Los ordenes de convergencia de cada esquema iterativo, $p_1 = 2$ y $p_2 = 3$ se multiplican y el método compuesto, alcanza orden $p_1 \cdot p_2$, en este caso concreto, resulta de orden seis.

La composición de estos dos métodos, cuyos índices de eficiencia son $I_N = 2^{\frac{1}{2}}$ y $I_H = 3^{\frac{1}{3}}$, Newton y Halley respectivamente, dan como resultado un método de dos pasos con un índice de eficiencia $I_{NH} = 6^{\frac{1}{5}}$. En este caso, el índice de eficiencia resultante, es menor que el de uno de los métodos, el de Halley, por tanto la composición no está justificada en este caso. Esto se debe a que estamos añadiendo más evaluaciones funcionales en cada iteración. Para evitar añadir un gran número de evaluaciones funcionales, usaremos la técnica de funciones peso, como se aplica en [46, 47, 48]. Con esta técnica, se reduce el número de evaluaciones funcionales, sin penalizar al orden de convergencia, ni por tanto, la eficiencia. El método el método de Traub [8], o de Potra-Pták, con orden de convergencia tres, su expresión iterativa comienza con el método de Newton, y continúa en un segundo paso, volviendo a aplicar Newton pero manteniendo la derivada del primer paso (derivada congelada) y evaluando la función en y_k, del primer paso

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(y_k)}{f'(x_k)} \qquad k = 0, 1, 2, \ldots. \end{aligned} \tag{2.11}$$

Después se diseñarían los conocidos métodos de Jarratt (2.8), y de Ostrowski [49], que está incluido en la familia King (2.2), en el caso de que $\beta = -2$,

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\
x_{k+1} &= y_k - \frac{f(x_k) + (2+\beta) f(y_k)}{f(x_k) + \beta f(y_k)} \frac{f(y_k)}{f'(x_k)} \qquad k = 0, 1, 2, \ldots .
\end{aligned} \tag{2.12}
$$

Los métodos (2.8) y (2.12) con dos pasos, y sin evaluar derivadas de la función de orden superior a uno, consiguen cuarto orden de convergencia. Son por tanto más eficientes que los métodos de Kung y Traub.

Chun [50] modifica el esquema de Traub incluyendo como factor a la función peso real G, suficientemente diferenciable en el entorno de cero, resultando el siguiente método

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\
x_{k+1} &= y_k - G(t_k) \frac{f(y_k)}{f'(x_k)} \qquad k = 0, 1, 2, \ldots
\end{aligned} \tag{2.13}
$$

donde $t_k = \dfrac{f(y_k)}{f(x_k)}$, alcanzado así el cuarto orden de convergencia. Esta técnica resulta muy útil para acelerar esquemas multipunto sin añadir evaluaciones funcionales y será empleada en el diseño de nuestras familias de métodos iterativos.

2.2. Conceptos de métodos iterativos vectoriales

El objetivo de estos métodos es aproximar raíces de sistemas de ecuaciones no lineales del tipo $F(x) = 0$, donde F es una función vectorial, $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, definida en un conjunto convexo D, con n ecuaciones no lineales y n incógnitas. El sistema se describe en cada ecuación por su función coordenada asociada, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, si $i = 1, 2, \ldots, n$

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) &= 0, \\ f_2(x_1, x_2, \ldots, x_n) &= 0, \\ & \vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0. \end{cases}$$

Normalmente, la solución de los sistemas no lineales no puede ser calculada analíticamente. Por este motivo, recientemente se está estudiando la forma de obtener soluciones aproximadas y fiables por medio de métodos iterativos, como se puede ver en [51, 52, 53, 54, 53, 55].

La literatura sobre métodos iterativos para resolver problemas escalares es numerosa. Sin embargo, los trabajos sobre problemas multidimensionales es aún escasa. En los últimos años se han publicado trabajos de métodos iterativos para resolver sistemas no lineales, haciendo modificaciones a métodos clásicos que aumentan la convergencia o disminuyen el número de evaluaciones funcionales y operaciones. Weerakoon y Fernando describen la modificación del método de Newton en [33], en [27] Hueso diseña una variante del método de Jarratt, en [56] Narang presenta una familia de orden seis, a partir del método de Chebyshev-Halley y Behl en [57] un método paramétrico.

En esta tesis presentamos una familia de métodos iterativos para resolver problemas no lineales con una función peso matricial en su esquema iterativo. Seleccionando un miembro de la familia, comparamos su eficiencia con métodos conocidos.

Estos esquemas generan una sucesión $\{x^{(k)}\}_{k\geq 0}$ se aproxima a la solución del sistema α partiendo de una aproximación $x^{(0)}$ próxima a la solución. Bajo ciertas condiciones, podemos asegurar que dicha sucesión es convergente, y la aproximaremos con la precisión deseada que requiere que exista este límite $\lim\limits_{x\to\infty} x^{(k)} = \alpha$ cumpliendo un criterio de error.

La solución del sistema se suele aproximar por métodos de punto fijo,

$$x^{(k+1)} = \Phi(x^{(k)}), \qquad k = 0, 1, \ldots$$

Estos esquemas se diferencian entre si por la forma en la que se define la función de iteración Φ.

Uno de los métodos más comúnmente utilizados para resolver sistemas de ecuaciones no lineales, es el método clásico de Newton, el cual alcan-

za una convergencia cuadrática, siempre que se cumplan determinadas condiciones, está definido por la siguiente expresión:

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \quad k = 0, 1, 2, ... \tag{2.14}$$

donde $F'(x^{(k)})$ es la matriz Jacobiana asociada a F evaluada en $x^{(k)}$.

Dependiendo de sus características, los métodos iterativos se clasifican de varias formas. Una de las más importantes es por el orden de convergencia, que refleja la velocidad con la que la sucesión converge a la solución. En [8] se encuentran definiciones importantes sobre estos conceptos para el caso vectorial.

Consideremos la sucesión $\{x^{(k)}\}_{k\geq 0} \subseteq \mathrm{R}^n$ generada por un método iterativo que converge a α. El método iterativo que genera esta sucesión tiene un orden de convergencia p, $p \geq 1$, si existe una constante positiva D tal que

$$\lim_{k\to\infty} \frac{||x^{(k+1)} - \alpha||}{||x^{(k)} - \alpha||^p} = D,$$

donde $D > 0$ se denomina constante de error asintótica. Si $p = 1$, $0 < D < 1$ podemos afirmar que el método tiene convergencia lineal.

En el caso multidimensional, denotamos por $e^{(k)} = x^{(k)} - \alpha$ el error cometido en la iteración k. Cualquier método vectorial cuyo orden de convergencia sea p satisface la siguiente relación entre el error en iteraciones consecutivas, que recibe el nombre de ecuación del error,

$$e^{(k+1)} = Le^{(k)^p} + \mathcal{O}(e^{(k)^{p+1}}),$$

donde L es una función p-lineal $L \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$. Nótese que $e^{(k)^p}$ es una notación que describe de forma compacta la p-tupla sobre la que actúa el operador p-lineal L, $(\mathrm{e}^{(k)}, e^{(k)}, \ldots, e^{(k)})$.

En esta memoria presentaremos dos familias de esquemas iterativos con convergencia de orden cuatro, para estimar raíces simples y múltiples de ecuaciones no lineales y un método para estimar soluciones no singulares de sistemas de ecuaciones no lineales, de orden seis.

En las pruebas numéricas se usan aproximaciones al orden de convergencia. Una de las primeras estimaciones del orden de convergencia

computacional fue publicado por Weerakoon y Fernando en [33], y expresado como

$$p \approx COC = \frac{\ln\left(||x^{(k+1)} - \alpha||/||x^{(k)} - \alpha||\right)}{\ln\left(||x^{(k)} - \alpha||/||x^{(k-1)} - \alpha||\right)}, \tag{2.15}$$

donde $x^{(k+1)}$, $x^{(k)}$ y $x^{(k-1)}$ son tres aproximaciones consecutivas a la raíz α, obtenidas del proceso de iteración.

Debido a que la raíz α, en la mayoría de los casos, no se conoce, Cordero y Torregrosa definen en [34] el orden de convergencia computacional aproximado (ACOC), el cual tiene la siguiente expresión para sistemas

$$p \approx ACOC = \frac{\ln\left(||x^{(k+1)} - x^{(k)}||/||x^{(k)} - x^{(k-1)}||\right)}{\ln\left(||x^{(k)} - x^{(k-1)}||/||x^{(k-1)} - x^{(k-2)}||\right)}. \tag{2.16}$$

Al diseñar métodos iterativos, el principal objetivo es la eficiencia, debido a que a mejor eficiencia, menor número de evaluaciones funcionales se utilizan, para alcanzar mayor orden de convergencia.

Para aumentar el orden de convergencia, diseñaremos métodos con varios pasos o multipunto de la forma

$$\begin{aligned}
y_1^{(k)} &= \Psi_1(x^{(k)}),\\
y_2^{(k)} &= \Psi_2(x^{(k)}, y_1^{(k)}),\\
&\vdots\\
y_{m-1}^{(k)} &= \Psi_{m-1}(x^{(k)}, y_1^{(k)}, \ldots, y_{m-2}^{(k)}),\\
x^{(k+1)} &= \Phi(x^{(k)}, y_i^{(k)}), \quad i = 1, 2, \ldots, m-1, \ldots
\end{aligned}$$

con la técnica de composición de métodos ya existentes, como en [58].

En [10] se demuestra que el orden de convergencia de un método multipaso es p, siendo $p = p_1 p_2 p_{m-1} p_m$, donde p_i es el orden de convergencia del método Ψ_i para $i = 1, \ldots, m-1$ y p_m es el orden de convergencia del método Φ.

Existen métodos sobradamente conocidos como los de Traub [8] y la familia King [24].

Por último, otra categoría con la que se clasifican los métodos iterativos, es si contienen Jacobianas en su expresión iterativa o, por el

contrario, son libres de Jacobianas. Para las demostraciones de métodos iterativos vectoriales se usa la siguiente notación, que se presenta en [59].

Sea $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ suficientemente diferenciable en D. La q-ésima derivada de F en $u \in \mathbb{R}^n$, $q \geq 1$, es la función q-lineal $F^{(q)}(u) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ tal que $F^{(q)}(u)(v_1 \ldots, v_q) \in \mathbb{R}^n$.

La q-ésima derivada de F satisface las siguientes propiedades:

- $F^{(q)}(u)(v_1, \ldots, v_{q-1}) \in \mathcal{L}$, donde $\mathcal{L}$ es el conjunto de operadores lineales de $\mathbb{R}^n$.
- $F^{(q)}(u)(v_{\sigma_1}, \ldots, v_{\sigma_q}) = F^{(q)}(u)(v_1, \ldots, v_q)$ para cada permutación σ de $1, 2, \ldots, q$.
- $F^{(q)}(u)(v_1, \ldots, v_q) = F^{(q)}(u) \cdot v_1, \ldots, v_q$.
- $F^{(q)}(u)v^{q-1}F^{(p)}v^p = F^{(q)}(u)F^{(p)}(u)v^{q+p-1}$.

Dado $\alpha+h \in \mathbb{R}^n$ en un entorno de la solución α, se puede aplicar el desarrollo de Taylor sobre la derivada entorno a α y asumir que la matriz Jacobiana de $F'(\alpha)$ no es singular, obteniendo

$$F(\alpha + h) = F'(\alpha) \left[h + \sum_{q=2}^{p-1} C_q h^q + O(h^p) \right], \tag{2.17}$$

donde $C_q = \dfrac{1}{q!}[F'(\alpha)]^{-1}F^{(q)}(\alpha)$, $q \geq 2$. Notemos que $C_q h^q \in \mathbb{R}^n$ ya que $F^{(q)} \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ y $F'(\alpha)^{-1} \in \mathcal{L}(\mathbb{R}^n)$.

Con el fin de comparar los métodos iterativos vectoriales, se utiliza el índice de eficiencia $EI = \rho^{\frac{1}{d}}$, presentado por Ostrowski en [7], siendo ρ el orden de convergencia y d es la cantidad de evaluaciones funcionales por iteración. Nótese que se deben evaluar n funciones escalares por cada F y n^2 por cada F'. Otro índice para comparar esquemas iterativos es el índice computacional de eficiencia (CE), presentado en [32]

$$CE = p^{\frac{1}{(d+op)}},$$

donde *op* es el número de productos/cocientes por iteración.

Se recuerda que el número de productos/cocientes que se necesitan para resolver un sistema lineal, mediante la eliminación de Gauss, es $\frac{1}{3}n^3+n^2-\frac{1}{3}n$ donde n es el tamaño del sistema.

El CE del método de Newton es

$$2^{1/\tau},\ \tau = \frac{1}{3}n^3 + 2n^2 + \frac{2}{3}n,$$

y es usado para compararlo con índices de otros esquemas iterativos.

2.3. Dinámica compleja

La estimación inicial de la que partimos para iterar en métodos para ecuaciones y sistemas de ecuaciones no lineales, y su relevancia a la hora de probar la convergencia de los esquemas iterativos ha centrado la atención en los últimos años de los investigadores. Para comprobar la estabilidad de un método, usaremos técnicas de dinámica compleja [58, 60, 61]. Cuando tenemos una familia de métodos, con estas herramientas, podemos predecir el comportamiento caótico o no, de cada miembro de la familia. En otros campos de las matemáticas, únicamente se centran en la solución analítica de los problemas, con los sistemas dinámicos podemos conocer el comportamiento de los métodos, antes de su ejecución.

Existen dos tipos de sistemas dinámicos, los sistemas dinámicos contínuos (SDC) y discretos (SDD). Los SDC están asociados a ecuaciones diferenciales, ordinarias y parciales, y los discretos, que son los que se utilizan en esta memoria, asociados a los métodos iterativos.

Los sistemas dinámicos discretos, son de la forma $x_{k+1} = \Phi(x_k)$. El ejemplo por antonomasia es el conjunto de Mandelbrot, que ha sido estudiado [62, 63].

A continuación definiremos conceptos que serán útiles para el análisis de la estabilidad del método, al crear aplicaciones en la esfera de Riemann $\hat{\mathbb{C}}$ (planos dinámicos, planos de parámetros) [64], con los que visualizermos el comportamiento del método, hacia la solución. El análisis dinámico de un método iterativo parte de aplicar un operador de punto fijo de los métodos iterativos sobre polinomios no lineales, calculando los puntos fijos (raíces o no de la ecuación), así como en las cuencas de atracción asociadas a los mismos.

2.3.1. Fundamentos

Se van a presentar los conceptos básicos que sirven para realizar el estudio dinámico complejo.
Sea $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, una función racional sobre la esfera de Riemann [22], $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Entonces $R(z) = \dfrac{P(z)}{Q(z)}$, con $P(z)$ y $Q(z)$ polinomios complejos, sin factores comunes. Dicha función racional se obtendrá aplicando el método iterativo bajo análisis sobre un polinomio de grado bajo.

La órbita de un punto $z_0 \in \hat{C}$ es definida como la sucesión de la aplicación del operador R en este punto [61, 65], determinada por el conjunto

$$\{z_0, R(z_0), R^2(z_0), \ldots, R^k(z_0), \ldots\},$$

donde $R^k(z_0)$ es el operador resultante de aplicar R sobre z_0, k veces.

Un punto $z^F \in \hat{\mathbb{C}}$ es un punto fijo cuando $R(z^F) = z^F$. La característica dinámica de los puntos fijos se clasifica según el valor de $\left|R'(z^F))\right|$, de esta manera $z^F \in \tilde{\mathbb{C}}$ es

- Atractor $\left|R'(z^F))\right| < 1$,
- Superatractor $\left|R'(z^F))\right| = 0$,
- Repulsor $\left|R'(z^F))\right| > 1$,
- Parabólico o neutral o indiferente, si $\left|R'(z^F))\right| = 1$.

La situación ideal es que los puntos fijos del operador racional R coincidan con las raíces del polinomio sobre el que se aplica. Si esto no ocurre, a esos puntos fijos se les denomina, puntos fijos extraños.

Los puntos periódicos se definen de forma similar que los fijos. Un punto z es periódico, de periodo T si

$$R^T(z) = z, R^k(z) \neq z,$$

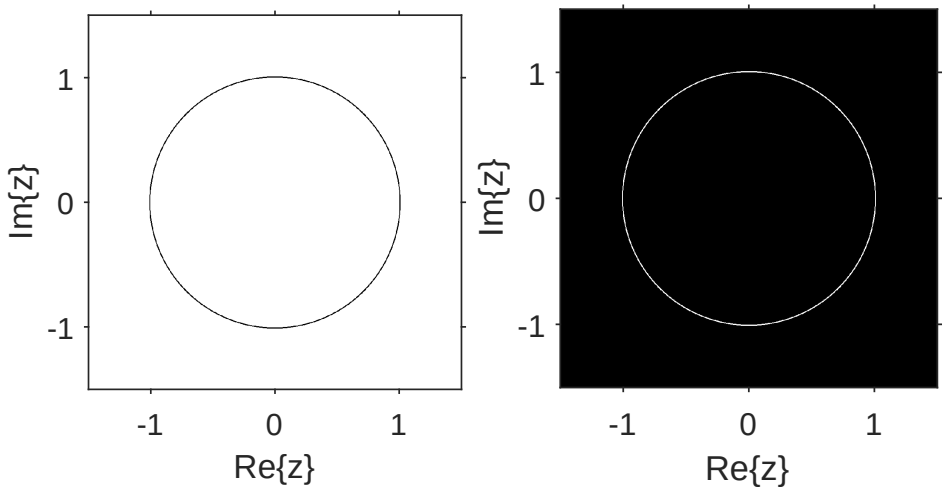
para todo $k < T$.

Los puntos críticos del operador, denominados z^C, son los que cumplen $R'(z^C) = 0$ [66], y si no coinciden con las raíces del polinomio, se denominan puntos críticos libres. Utilizaremos estos puntos para representar los planos de parámetros.

Las cuencas de atracción, determinan el estado final de la órbita de cualquier punto, del plano complejo, tras la sucesiva aplicación del operador R. De esta forma, definimos la cuenca de atracción de un punto fijo atractor $z^F \in \hat{\mathbb{C}}$ como el conjunto de preimágenes de cualquier orden, que cumplen

$$\mathcal{A}(z^F) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z^F, n \to +\infty\}.$$

En relación con las cuencas de atracción tenemos los conjuntos Fatou y Julia [67, 68]. El conjunto de Fatou $\mathcal{F}(R)$, se conoce como el conjunto de puntos que tiende a un punto atractor z^*. Su conjunto complementario en la esfera de Riemann es el de Julia $\mathcal{J}(R)$, el cual contiene a los puntos repelidos, delimitando las fronteras entre las cuencas de atracción.



(a) Conjunto de Julia (b) Conjunto de Fatou

Figura 2.1: Conjuntos de Julia (a) y de Fatou (b)

El más sencillo de obtener es el del método de Newton sobre el polinomio $p(z) = (z-a)(z-b)$. Tras aplicar la transformación de Möbius [69], el operador racional que queda es $R(z) = z^2$. El conjunto de Julia es precisamente la circunferencia $|z| = 1$, mientras que el conjunto de Fatou es el resto del plano complejo, como se puede ver en la Figura 2.1.

El siguiente resultado relaciona los puntos críticos con las cuencas de atracción [68, 67].

Teorema 1 *(Fatou-Julia) Sea R una función racional. La cuenca de atracción inmediata de cada punto atractor, contiene al menos, un punto crítico.*

Así, los puntos críticos pueden pertenecer a cuencas de atracción de órbitas periódicas atractoras, y no converger a raices de $p(z)$. De este modo, calcular los puntos críticos del sistema es fundamental.

2.3.2. Clases de conjugación

Hasta ahora se han descrito los conceptos dinámicos, definidos sobre el operador racional que resulta de aplicar el método iterativo sobre un polinomio $p(z)$. Con las clases de conjugación se simplifica el estudio dinámico, generalizando el análisis sobre funciones particulares.

A continuación mostramos algunos de los resultados clásicos empleados.

Teorema 2 *(Teorema del escalado,[23]) Sea $f(z)$ una función analítica en y $A(z) = \delta z + \gamma$, con $\delta \neq 0$, una transformación afín. Sea $g(z) = \theta(f \circ A)(z)$, con $\theta \neq 0$, entonces el operador de punto fijo R_f es conjugado analíticamente a R_g por A, es decir,*

$$(A \circ R_g \circ A^{-1})(z) = R_f(z).$$

Teorema 3 *Sea $q(z) = a_1 z^2 + a_2 z + a_3, a_1 \neq 0$ un polinomio cuadrático cualquiera, con raíces simples, y R_f y R_g operadores racionales. Entonces $q(z)$ se puede reducir a $p(z) = z^2 + c$, donde $c = 4a_1a_3 - a_2^2$. Esta transformación afín produce una conjugación entre los operadores R_f y R_g, correspondientes a los polinomios $p(z)$ y $q(z)$, respectivamente.*

Los operadores asociados a los esquemas iterativos que satisfacen el Teorema del Escalado simplifican el estudio dinámico porque solo será necesario estudiar un operador racional más sencillo, pudiendo generalizar los resultados a cualquier miembro. De esta forma es posible conocer la estabilidad de un método para un conjunto de funciones.

Mostramos un ejemplo de las clases de conjugación, en el caso del método de Newton, aplicado sobre polinomios cuadráticos $p_c(z) = z^2 + c$, el operador racional asociado al método de Newton sobre $p_c(z)$ depende del parámetro c, resultando

$$R_{p_c}(z) = \frac{z^2 - c}{2z}.$$

Podemos eliminar el parámetro c utilizando la transformación de Möbius

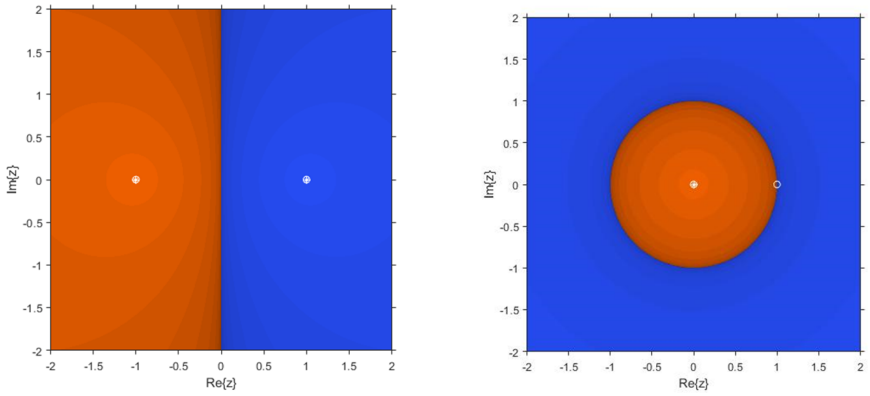
$$h(z) = \frac{z - i\sqrt{c}}{z + i\sqrt{c}}$$

que cumple con las propiedades $h(\infty) = 1$, $h(i\sqrt{c}) = 0$ y $h(-i\sqrt{c}) = \infty$. Esta transformación lleva una de las raíces al cero, la otra raíz al infinito, y la divergencia al 1.

Después de la conjugación, el operador asociado al esquema iterativo de Newton es

$$R_h(z) = (h \circ R_{p_c} \circ h^{-1}) = z^2,$$

quedando los conjuntos de Fatou y Julia distribuidos en el plano complejo tal y como se observa en la Figura 2.2.



(a) Sin transformación Möbius (b) Con transformación Möbius

Figura 2.2: Cuencas de atracción del método de Newton sobre $p(z) = z^2 - 1$

Este tipo de representación recibe el nombre de plano dinámico, el cual definimos a continuación.

2.3.3. Planos Dinámicos

Para estudiar el comportamiento de un método iterativo, existen dos herramientas gráficas, utilizadas en esta tesis, el plano dinámico y el plano de parámetros. El plano dinámico permite elegir estimaciones iniciales que convergen a puntos fijos atractores, mientras que el plano de parámetros,

ayuda a seleccionar los miembros de una familia de métodos iterativos con mayor estabilidad.

El plano dinámico determina, para un conjunto de estimaciones iniciales en una determinada zona del plano complejo, las cuencas de atracción de puntos fijos atractores y puntos fijos atractores extraños. Representaciones utilizadas en [70, 71, 72, 73]. Cada punto fijo atractor se representa con colores diferentes, el eje de abcisas representa la parte real y el de ordenadas la parte imaginaria. Cada pareja de puntos del mallado representa un número complejo que es la estimación inicial para comenzar a iterar con el método. Dependiendo hacia donde converga cada estimación inicial, estará coloreada de un color u otro.

La Figura 2.3 muestra los planos dinámicos del método de Newton cuando se aplica sobre el polinomio $p(z) = z^4 - 1$. Este polinomio, tiene cuatro raíces, que coinciden con los únicos puntos fijos atractores, que se encuentran en $z^F = z^* \in \{1, -1, i, -i\}$.

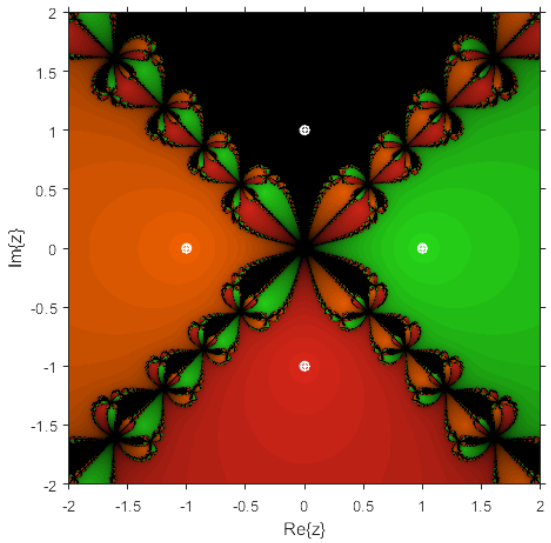


Figura 2.3: Plano dinámico, del método de Newton, al aplicarlo sobre $p(z) = z^4 - 1$

El código desarrollado en MATLAB® que representa planos dinámicos se encuentra en [64].

2.3.4. Planos de Parámetros

Los planos de parámetros se representan dividiendo el plano complejo eun un mallado donde el eje de abcisas corresponde a los número reales y el eje de ordenadas el de los números imaginarios. Cada punto de este plano, es un valor del parámetro, esto es, un miembro diferente de una familia de métodos. Existen tantos planos de parámetros, como puntos críticos libres tenga el operador de la familia. En el plano de parámetros encontramos puntos de dos colores diferentes, puntos rojos, si el método que representa converge a algún punto fijo atractor del operador, y puntos blancos, en otro caso. De esta forma se eligen los miembros de una familia de métodos, con mayor estabilidad.

En la Figura 2.4 se representa el plano de parámetros, correspondiente al método de Newton amortiguado

$$x_{k+1} = x_k - \gamma \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, ... \tag{2.18}$$

donde $\gamma \neq 0$, $\gamma \in \mathbb{R}$. La región roja del plano de parámetros representa los valores complejos del parámetro (y por tanto los métodos de la familia) que convergen a las raíces del polinomio original, con orden de convergencia cuadrático, esto sucede cuando $\gamma = 1$, para el resto de valores de γ, los puntos de color negro, son métodos que convergen a puntos atractores diferentes a las raíces y con convergencia lineal. Se ha representado con un mallado de 500×500 puntos y 200 iteraciones.

Para representar los planos de parámetros, el plano complejo se divide en un mallado de valores complejos de γ, siendo los ejes de abcisas y de ordenadas las partes real e imaginaria de éste, respectivamente. Cada punto del plano se corresponde con un valor de γ, y por tanto, con un método de la clase de métodos iterativos. Para cada valor de γ, se itera sucesivamente el método correspondiente tomando como estimación inicial un punto crítico libre del operador, de manera que existen tantos planos de parámetros como puntos críticos libres independientes (bajo conjugación analítica) tenga el operador asociado a la familia. Los puntos del plano se representan en rojo cuando el método asociado converge a alguno de los puntos fijos atractores del operador, y en negro en cualquier otro caso. Esta herramienta gráfica permite seleccionar los valores del parámetro que dan lugar a los métodos más estables de la familia.

En la Figura 2.4 se muestra el plano de parámetros de la función racional asociada al método de Newton amortiguado sobre $p(z) = (z-a)(z-b)$ tras

la transformación de Möbius. Se ha empleado un mallado de 500×500. Se representan en rojo los métodos de la familia que convergen a alguna de las raíces del polinomio, determinando los métodos más estables de la familia.

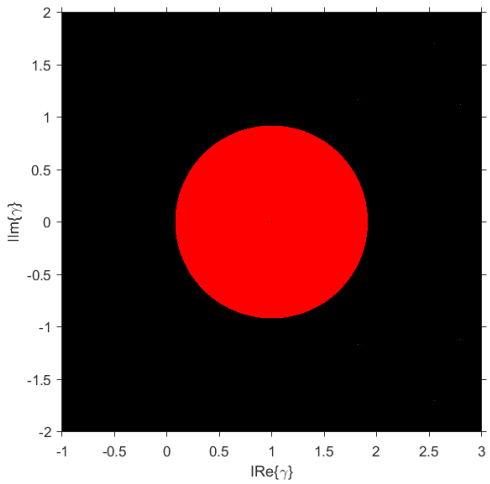


Figura 2.4: Plano de parámetros de Newton amortiguado sobre polinomios cuadráticos

Capítulo 3

Artículos científicos

3.1. Articulo 1

En este artículo se presenta una familia paramétrica de métodos iterativos, de cuarto orden de convergencia. También se incluye su estudio dinámico, con el que conseguimos representar las cuencas de atracción y el plano de parámetros, que permiten elegir la mejor estimación inicial, y los miembros de la familia más eficientes, respectivamente. Este artículo se presentó en el congreso Modelling for Engineering & Human Behaviour 2021 (Valencia) y se publicaron algunos resultados parciales en las actas del mismo congreso, con ISBN: 978-84-09-36287-5. Por otra parte una aplicación de este método a un problema constitutivo del hormigón armado, se expuso en el Congreso de Métodos Numéricos en Ingeniería, en septiembre de 2022 (Las Palmas de Gran Canaria).

fractal and fractional

MDPI

Article

Parametric Family of Root-Finding Iterative Methods: Fractals of the Basins of Attraction

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Abstract: Research interest in iterative multipoint schemes to solve nonlinear problems has increased recently because of the drawbacks of point-to-point methods, which need high-order derivatives to increase the order of convergence. However, this order is not the only key element to classify the iterative schemes. We aim to design new multipoint fixed point classes without memory, that improve or bring together the existing ones in different areas such as computational efficiency, stability and also convergence order. In this manuscript, we present a family of parametric iterative methods, whose order of convergence is four, that has been designed by using composition and weight function techniques. A qualitative analysis is made, based on complex discrete dynamics, to select those elements of the class with best stability properties on low-degree polynomials. This stable behavior is directly related with the simplicity of the fractals defined by the basins of attraction. In the opposite, particular methods with unstable performance present high-complexity in the fractals of their basins. The stable members are demonstrated also be the best ones in terms of numerical performance of non-polynomial functions, with special emphasis on Colebrook-White equation, with wide applications in Engineering.

Keywords: nonlinear problems; iterative methods; weight functions; complex dynamics; basin of attraction; fractal

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1. Introduction

The solution of nonlinear equations is a typical problem in engineering and experimental sciences. Problems with gases, liquids, and mechanics require calculating the roots of the equations with the iterative. Among Chemistry problems needing to solve this kind of equations are chemical equilibrium problems, global reaction rates in packed bed reactors [1], radioactive transfer [2], continuous stirred tank reactors (see [3]) or to simulate flow transport in a pipe [4]. In general, these equations cannot be solved analytically and we must resort to iterative methods to approximate their solutions. With the increasing speed of computers, numerical techniques have become indispensable for scientists and engineers. The principle of these methods is to approach the solution x^* of a nonlinear equation of the form $f(x) = 0$, through a sequence of iterations, starting from an initial estimation of x_0. The most known and widely used method to solve nonlinear equations is Newton's scheme, whose iterative expression is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k = 0, 1, 2, \ldots \quad (1)$$

and it presents the quadratic convergence. This method requires two functional evaluations $(d = 2)$, one of the function and one of its first derivative, per iteration. Kung-Traub's conjecture [5] states that an iterative method without memory for finding a simple zero of an scalar equation is optimal if its order of convergence is equal to 2^{d-1}. Therefore, Newton's method is optimal.

Many variants of Newton's scheme have been constructed by means of several techniques, setting different multistep schemes. Some of them use Adomian decomposition (see [6], for instance). Another kind of iterative schemes are those of using higher-order derivatives, for example in Chebyshev-type methods by some approximatio [7]. Moreover, the direct composition of known methods with a later treatment to reduce the number of functional evaluations is often used in order to generate new methods. Indeed, a weight-function procedure has been used recently to increase the convergence order of known methods [8,9], in order to obtain optimal schemes.

Moreover, it is known that the sensitiveness of a scheme to the initial seeds increases as well as does the order [10]. It is widely accepted that the dynamical behavior of the rational function related to an iterative scheme provides us with important information about its stability and reliability [11]. In these terms, Amat et al. in [12] described the dynamical performance of some known families of iterative methods. More recently, in [9,13–17], different authors analyze the qualitative behavior of several known methods or classes of iterative schemes. Most of these studies demonstrate some elements with very stable behavior, which is proven to be useful in practice, and also different pathological performances, such as attracting fixed points different from the solution of the problem, periodic orbits, etc. The key tool to understanding the behavior of the different members of the family are the parameter planes.

We aim to design a new multipoint fixed point class, without memory, that improves or brings together the existing ones, as for example appears in [18,19], in different areas such as computational efficiency, stability and convergence order. There exists a wide number of optimal iterative methods with different orders of convergence, such as [15]; some of them can be grouped as special cases of our proposed class of iterative procedures.

Let us search the conditions that parameter γ and G function, from the iterative expression

$$\begin{aligned} y_k &= x_k - \gamma \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - G(\eta_k) \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{2}$$

must meet to reach the fourth convergence order, where the variable of the weight function is $\eta_k = \dfrac{f'(y_k)}{f'(x_k)}$.

The rest of the paper is organized as follows: in Section 2, we prove the fourth-order convergence of the proposed class of iterative methods, under some conditions. Moreover, a particular subclass is presented and some known methods have been found as special cases of our proposed family. Section 3 is devoted to the qualitative study of our proposed parametric class of iterative schemes, giving rise to some stable and unstable members, that are numerically checked in Section 4, along with other known methods. These numerical tests demonstrate the good performance in both academical and applied problems, such as the Colebrook-White equation.

2. Convergence Analysis

In the next result, we present the sufficient conditions of the weight function G and parameter γ that guarantee the convergence of the proposed class.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open interval I and $x^* \in I$ a simple root of equation $f(x) = 0$. Let $G(\eta)$ be a real function satisfying $|G'''(1)| < +\infty$,*

$G''(1) = \frac{9}{4}$, $G'(1) = -\frac{3}{4}$ *and* $G(1) = 1$. *If* $\gamma = \frac{2}{3}$ *and we choose an initial approximation* x_0 *close enough to* x^*, *then the family of iterative methods defined by*

$$\begin{aligned} y_k &= x_k - \gamma \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - G(\eta) \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{3}$$

satisfies the error equation below:

$$e_{k+1} = \left[\left(5 + \frac{32G'''(1)}{81} \right) c_2^3 - c_2 c_3 + \frac{c_4}{9} \right] e_k^4 + O(e_k^5),$$

where $c_k = \frac{1}{k!} \frac{f^{(k)}(x^*)}{f'(x^*)}$, $k = 2, 3, \ldots$, $\eta_k = \frac{f'(y_k)}{f'(x_k)}$ *and* $e_k = x_k - x^*$, $k = 1, 2, \ldots$. *Therefore, all the members of class* (3) *converges to* x^* *with an order of convergence four.*

Proof. Let x^* be a simple zero of f. As f is a sufficiently differentiable function, using the Taylor expansion for $f(x_k)$ and $f'(x_k)$ at about x^*, we obtain

$$f(x_k) = f'(x^*)[e_k + c_2 e_k^2 + c_3 e_k^3 + c_4 e_k^4] + \mathcal{O}(e_k^5),$$

and

$$f'(x_k) = f'(x^*)[1 + 2c_2 e_k + 3c_3 e_k^2 + 4c_4 e_k^3] + \mathcal{O}(e_k^4).$$

From these expressions, we obtain the first step

$$y_k = (1 - \gamma) e_k + \gamma c_2 e_k^2 - 2(\gamma(c_2^2 - c_3)) e_k^3 + \gamma(4c_2^3 - 7c_2 c_3 + 3c_4) e_k^4 + \mathcal{O}(e_k^5).$$

Expanding in Taylor's series $f(y_k)$, around x^*,

$$f'(y_k) = f'(x^*) \Big[1 - 2((-1 + \gamma)c_2) e_k + (2\gamma c_2^2 + 3(-1 + \gamma)^2 c_3) e_k^2 + 2(-2\gamma c_2^3 + (5 - 3\gamma)\gamma c_2 c_3 - 2(-1 + \gamma)^3 c_4) e_k^3 \Big] + \mathcal{O}(e_k^4).$$

and then, combining these expressions,

$$\eta = 1 - 2(\gamma c_2) e_k + 3\gamma(2c_2^2 + (-2 + \gamma)c_3) e_k^2 - 4(\gamma(4c_2^3 + (-7 + 3\gamma)c_2 c_3 + (3 - 3\gamma + \gamma^2)c_4)) e_k^3 + \mathcal{O}(e^4). \tag{4}$$

Let us represent function G by its Taylor polynomial of the third order around 1, as η tends to 1 when k tend to infinity:

$$\begin{aligned} G(\eta) = {} & G(1) - 2(\gamma G'(1) c_2) e_k + \gamma(2(3G'(1) + \gamma G''(1)) c_2^2 + 3(-2 + \gamma) G'(1) c_3) e_k^2 \\ & + \Big(\frac{-4}{3} \gamma(12G'(1) + \gamma(9G''(1) + \gamma G'''(1))) c_2^3 - 2\gamma(2(-7 + 3\gamma)G'(1) + 3(-2 + \gamma)\gamma G''(1)) c_2 c_3 \\ & - 4\gamma(3 - 3\gamma + \gamma^2) G'(1) c_4 \Big) e_k^3 + \mathcal{O}(e_k^4). \end{aligned}$$

Therefore,

$$\begin{aligned} e_{k+1} = {} & (1 - G(1)) e_k + (G(1) + 2\gamma G'(1)) c_2 e_k^2 + (-2(G(1) + \gamma(4G'(1) + \gamma G''(1)))) c_2^2 + ((2G(1) - 3(-2 + \gamma)\gamma G'(1)) c_3) e_k^3 \\ & + \Big(\Big(4G(1) + 26\gamma G'(1) + 14\gamma^2 G''(1) + \frac{4\gamma^3 G'''(1)}{3} \Big) c_2^3 + (-7G(1) + \gamma((-38 + 15\gamma)G'(1) + 6(-2 + \gamma)\gamma G''(1))) c_2 c_3 \\ & + (3G(1) + 4a(3 - 3a + a^2) G'(1)) c_4 \Big) e_k^4 + \mathcal{O}(e_k^5). \end{aligned}$$

To achieve the order of convergence four, it is necessary to force the coefficients of e_k^p, $p = 1, 2, 3$ to be null. Then, we obtain that the following conditions are needed: $\gamma = \dfrac{2}{3}$, $G''(1) = \dfrac{9}{4}$, $G'(1) = \dfrac{-3}{4}$ and $G(1) = 1$.

By substituting them in the error equation, we obtain that it is a function of $G'''(1)$,

$$e_{k+1} = \left[\left(5 + \frac{32G'''(1)}{81}\right)c_2^3 - c_2c_3 + \frac{c_4}{9}\right]e_k^4 + \mathcal{O}(e_k^5),$$

and $|G'''(1)| < +\infty$ must be satisfied. □

Some known schemes can be found as particular cases of family (3) satisfying all the conditions of Theorem 1. Firstly, the well-known Jarratt's method [20], with the iterative expression

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \frac{1}{2}\frac{3f'(y_k) + f'(x_k)}{3f'(y_k) - f'(x_k)}\frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, ..., \end{aligned}$$

where $G(\eta) = \dfrac{3\eta + 1}{6\eta - 2}$.

Moreover, the method designed by Hueso et al. in [21], with iterative expression

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \left(-\frac{1}{2} + \frac{9}{8}\frac{f'(x_k)}{f'(y_k)} + \frac{3}{8}\frac{f'(y_k)}{f'(x_k)}\right)\frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots, \end{aligned}$$

where $G(\eta) = -\dfrac{1}{2} + \dfrac{9}{8\eta} + \dfrac{3}{8}\eta$ and the scheme from Khattri and Abbasbandi constructed in [22], defined as

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \left(1 + \frac{21}{8}\frac{f'(y_k)}{f'(x_k)} - \frac{9}{8}\left(\frac{f'(y_k)}{f'(x_k)}\right)^2 + \frac{15}{8}\left(\frac{f'(y_k)}{f'(x_k)}\right)^3\right)\frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots \end{aligned}$$

are particular members of the most general proposed class of iterative methods (3) with $G(\eta) = 1 + \dfrac{21}{8}\eta - \dfrac{9}{2}\eta^2 + \dfrac{15}{8}\eta^3$.

On the other hand, by using the conditions deduced in the previous result, we select a particular subclass of iterative methods, depending on a parameter α, whose iterative expression is

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \left(1 - \frac{3}{4}(\eta - 1) + \frac{9}{8}(\eta - 1)^2 + \alpha(\eta - 1)^3\right)\frac{f(x_k)}{f'(x_k)}, \ k = 0, 1, 2, \ldots \end{aligned} \tag{5}$$

Let us remark that all the members of this family of iterative schemes have fourth-order of convergence, as they satisfy all the hypothesis of Theorem 1. The differences in their performance can be studied by using the tools of complex discrete dynamics; so, the wideness of the sets of converging initial estimations can be deduced depending on α.

3. Stability Analysis

In order to study the dynamical behaviour of the iterative methods described in (5), it is necessary to recall a few concepts. Let us consider a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The successive application of the operator R on one point $z_0 \in \hat{\mathbb{C}}$ is defined as the orbit of this point [12,23]:

$$\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\},$$

where $R^n(z_0)$ means R applied on z_0, n times. In our context, R is obtained by applying the class of iterative methods on a quadratic polynomial $p(z)$.

Thus, a fixed point $z^F \in \hat{\mathbb{C}}$ of R satisfies $R(z_F) = z_F$. It is worthy to notice that it is possible to find fixed points of R that are not roots of the polynomial; in this case, those points are called strange fixed points. The stability of fixed points is classified as follows:

- Attractive if $|R'(z^F)| < 1$.
- Parabolic or Neutral if $|R'(z^F)| = 1$.
- Repulsive if $|R'(z^F)| > 1$.
- Superattractive if $R'(z^F) = 0$.

On the other hand, the basins of attraction [24] determine the final state of the orbit of any point in the complex plane after successive application of operator R. We define the basin of attraction of a fixed point $z^F \in \hat{\mathbb{C}}$ as the set of preimages of any order that meets it:

$$\mathcal{A}(z^F) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z^F, n \to +\infty\}.$$

Moreover, the roots of the equation $R'(z_0) = 0$ are called critical points of operator R. Their asymptotic performance plays an important role in the stability of the method [25]. Moreover, in the connected component of the basin of attraction holding the attractor, there exists one critical point. Indeed, superattracting fixed points are also critical points; furthermore, critical points not being zeros of $p(z)$ are defined as free critical points.

On the other hand, the union of the basins of attraction defines the Fatou set of R. Its complementary set in $\widehat{\mathbb{C}}$ is called the Julia set.

In this section, we analyze the dynamical behavior of fourth-order parametric family (5) on the quadratic polynomial $p(z) = (z - a)(z - b)$, where $a, b \in \mathbb{C}$. So, a rational function $O_p(z)$ is obtained, depending on the parameter of the class, $alpha$, and also depending on the roots a and b. To obtain a simpler operator, as the fixed point operator satisfies the Scaling Theorem, we use the Möbius transformation [23]

$$M(z) = \frac{z - a}{z - b}, \qquad M^{-1}(z) = \frac{zb - a}{z - 1},$$

with properties:

$$M(\infty) = 1, \qquad M(a) = 0, \qquad M(b) = \infty,$$

that yields a rational function that, being conjugated to $O_p(z)$ (and therefore, with equivalent dynamical behavior), does no longer depend on a and b:

$$O_\alpha(z) = \left(M \circ O_p \circ M^{-1}\right)(z) = z^4 \frac{135 + 64\alpha + 378z + 378z^2 + 162z^3 + 27z^4}{2 + 162z + 378z^2 + 378z^3 + 135z^4 + 64\alpha z^4}.$$

By solving equation $O_\alpha(z) = z$, the fixed points of the rational function are obtained. Among them are $z = 0$ and $z = \infty$, coming from the roots of the polynomial previous to the Möbius map. The asymptotic behavior of all the fixed points plays a key role in the stability of the iterative methods involved, as the convergence to fixed points different from the roots means an important drawback for an iterative method; thus, we proceed below with this analysis.

A direct result of the Möbius transformation applied on this rational function is the conjugacy by the inverse,

$$\frac{1}{O_\alpha(z)} = O_\alpha\left(\frac{1}{z}\right).$$

The immediate consequences of this result are:

(a) If $O_\alpha(z^F) = z^F$, then $O_\alpha(1/z^F) = 1/z^F$.
(b) Except for some specific values of the α simplifying the operator, $z = 1$ is an strange fixed point of rational operator $O_\alpha(z)$.
(c) The stability function of two conjugate fixed points coincide,

$$O'_\alpha(z^F) = 1/O'_\alpha(z^F).$$

3.1. Performance of the Strange Fixed Points

The behavior of fixed points different from $z = 0$ and $z = \infty$ depends on α. In the following result, the stability of $z = 1$ is established.

Theorem 2. *$z = 1$ is an strange fixed point of $O_\alpha(z)$ if $\alpha \neq -\frac{135}{8}$. Thus, $z = 1$ is attracting if $\left|z + \frac{135}{8}\right| > 54$, parabolic or neutral when $\left|z + \frac{135}{8}\right| = 54$.*

Proof. The behavior of $z = 1$ is given by

$$\left|O'_\alpha(1)\right| = \left|\frac{432}{8\alpha + 135}\right|.$$

Thus,

$$\left|\frac{432}{8\alpha + 135}\right| \leq 1 \quad \text{is equivalent to} \quad 432 \leq |8\alpha + 135|.$$

Let us denote $\alpha = a + ib$. Then,

$$168399 < 16(135a + 4a^2 + 4b^2)$$

and

$$b^2 + (a + 135/8)^2 > 2916 = 54^2.$$

Therefore,

$$\left|O'_\alpha(1)\right| \leq 1 \text{ if and only if } \left|z + \frac{135}{8}\right| > 54.$$

Finally, if α satisfies $\left|z + \frac{135}{8}\right| < 54$, then $|O'_\alpha(1)| > 1$ and $z = 1$ is a repulsive point. It is clear that it is parabolic in the boundary $|O'_\alpha(1)| = 1$. □

In Figure 1, the stability function of $z = 1$ can be observed. It can be noticed that complex values of α inside the region $\left|z + \frac{135}{8}\right| < 54$ define fourth-order iterative schemes, whose numerical performance do not include divergence.

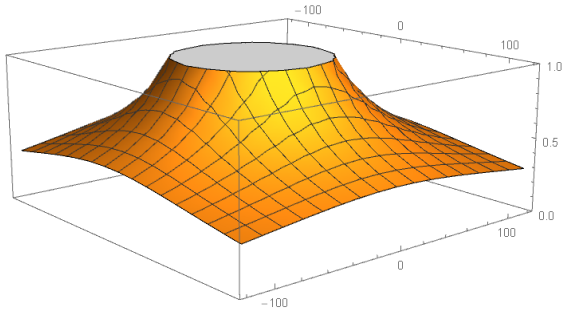
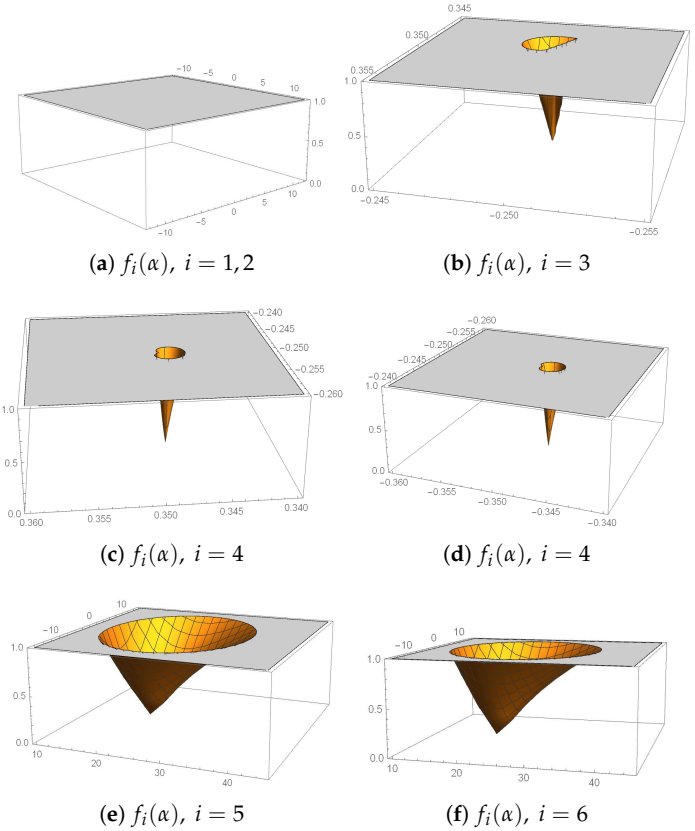


Figure 1. Stability region of $z = 1$.

The demonstration of this result is similar to those of Theorem 2. In Figure 2, the set of stability regions of strange fixed points $f_1(\alpha)$ to $f_6(\alpha)$ appear.

Proposition 1. *The roots of* $F(t) = 27 + 189t + 567t^2 + (810 - 64\alpha)t^3 + 567t^4 + 189t^5 + 27t^6$ *are strange fixed points of* $O_\alpha(z)$*, different from* $z = 1$*, and are denoted by* $f_i(\alpha)$*,* $i = 1, 2, \ldots, 6$*. These strange fixed points are reduced to four if* $\alpha = \frac{297}{8}$*, as* $f_1(\frac{297}{8}) = f_2(\frac{297}{8}) = 1$*. For* $\alpha \neq \frac{297}{8}$*,*

- $f_1(\alpha)$ *and* $f_2(\alpha)$ *are conjugate and repulsive, with independence of the value of parameter* α*.*
- $f_3(\alpha)$ *and* $f_4(\alpha)$ *are attractors for values of* α *in small regions of the complex plane, inside the complex area* $[-0.26, -0.24] \times [-0.36, -0.34]$ *and* $[-0.26, -0.24] \times [0.34, 0.36]$*. Moreover, both are superattracting for* $\alpha \approx -0.250121 \pm 0.348771i$*.*
- $f_5(\alpha)$ *and* $f_6(\alpha)$ *are conjugate and attractors for values of* α *inside the complex area* $[10, 40] \times [-15, 15]$*. Moreover, both are superattracting for* $\alpha \approx 20.3811$*.*



(a) $f_i(\alpha)$, $i = 1, 2$ **(b)** $f_i(\alpha)$, $i = 3$

(c) $f_i(\alpha)$, $i = 4$ **(d)** $f_i(\alpha)$, $i = 4$

(e) $f_i(\alpha)$, $i = 5$ **(f)** $f_i(\alpha)$, $i = 6$

Figure 2. Regions of stability corresponding to $f_i(\alpha)$, $i = 1, 2, \ldots, 6$.

3.2. Critical Points and Parameter Planes

In order to determine the critical points, we calculate the first derivative of $O_\alpha(z)$,

$$O'_\alpha(z) = \frac{108z^3(135(1+z)^8 + 32\alpha(1+z)^6(2-3z+2z^2))}{(27+162z+378z^2+378z^3+(135+64\alpha)z^4)^2}.$$

By definition, the roots of $O'_\alpha(z) = 0$ are called critical points. As the order of convergence of our class of proposed iterative methods is higher than two, those fixed points coming from the original roots of the polynomial, that is, $z = 0$ and $z = \infty$, are also critical points. In the next result, the rest of critical points are determined (called free critical points).

Proposition 2. *The number of free critical points of operator $O_\alpha(z)$ is:*

- *One, if $\alpha = 0$ or $\alpha = -\frac{135}{8}$. In these cases, the reduced rational operator is:*

$$\begin{aligned} O_{-\frac{135}{8}}(z) &= \frac{x^4\left(27x^4+162x^3+378x^2+378x-945\right)}{-945x^4+378x^3+378x^2+162x+27}, \\ O_0(z) &= \frac{x^4\left(27x^4+162x^3+378x^2+378x+135\right)}{135x^4+378x^3+378x^2+162x+27}, \end{aligned}$$

  whose only free critical point is $z = -1$, that is a pre-image of $z = 1$.
- *Three, if $\alpha \neq 0$ and $\alpha \neq -\frac{135}{8}$, as in this case, they are defined as:*

$$\begin{aligned} cr_1(\alpha) &= -1, \\ cr_2(\alpha) &= \frac{-135+48\alpha-4\sqrt{14}\sqrt{-135\alpha-8\alpha^2}}{135+64\alpha}, \\ cr_3(\alpha) &= \frac{-135+48\alpha+4\sqrt{14}\sqrt{-135\alpha-8\alpha^2}}{135+64\alpha}. \end{aligned}$$

It is easy to prove that $cr_2(\alpha) = \dfrac{1}{cr_3(\alpha)}$. Therefore, $cr_2(\alpha) = cr_3(\alpha) = -1$ when $-135\alpha - 8\alpha^2 = 0$.

As we have said, a classical result states that there is at least one critical point related with each basin of attraction. As $z = 0$ and $z = \infty$ are both superattracting fixed points of $O_\alpha(z)$, they also are critical points and give rise to their respective Fatou components. For the other critical points, we can establish the following remarks:

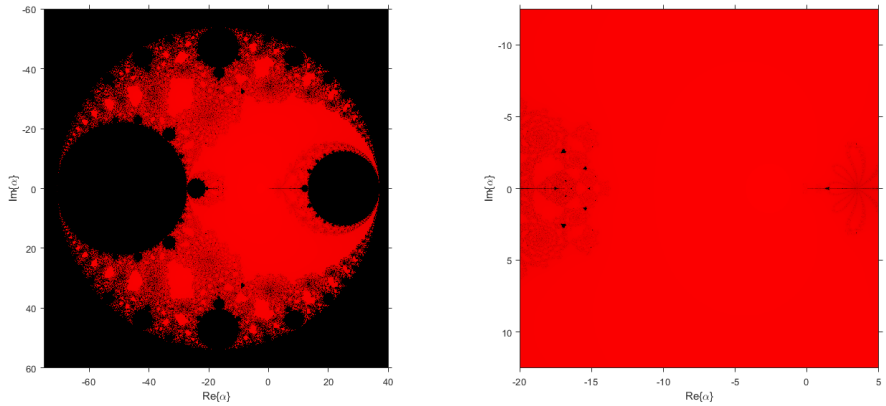
(a) If $\alpha = 0$, then $cr_2 = cr_3 = -1$, and it is a pre-image of the fixed point $z = 1$: $O_0(-1) = 1$. As $z = 1$ is repulsive for $\alpha = 0$, $z = -1 \in \mathcal{J}(O_\alpha)$. Thus, $O_p(z)$ has only two invariant Fatou components, $\mathcal{A}(0)$ and $\mathcal{A}(\infty)$.

(b) If $\alpha = -\dfrac{135}{8}$, then $cr_2 = cr_3 = -1$, and $O_{-\frac{135}{8}}(-1) = 1$. As $z = 1$ is not a fixed point when $\alpha = -\dfrac{135}{8}$, then $z = -1 \in \mathcal{J}(O_\alpha)$ and its orbit will remain at Julia set until the rounding error makes it fall into the basin of attraction of $z = 0$ or $z = \infty$.

(c) For the rest of the values of $\alpha \in \mathbf{C}$, we gave three critical points.

As we have previously stated, the dynamical performance of operator $O_\alpha(z)$ depends on the values of the parameter α. In Figure 3, we can observe the parameter space associated with family (5): each point of the parameter plane is associated with a complex value of α, i.e., with an element of family (5). A free critical point is employed as the starting point and, if for an specific value of α, this critical point converges to $z = 0$ or $z = \infty$, and then the point representing the value of α is painted in red color. Those values of α that make the critical point not converge to $z = 0$ or $z = \infty$ are painted in black color. Therefore, each connected component of the parameter plane gives us subfamilies of procedures of

class (5) with similar performance. The fractal defined as the boundary of these connected components separates regions of stable and unstable performance.



(**a**) $cr_2(\alpha)$ (**b**) $cr_2(\alpha)$, a detail

Figure 3. Parameter plane from free independent critical point.

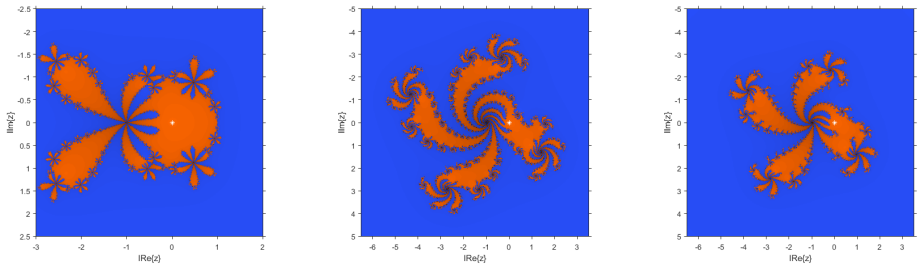
In Figure 3, the parameter plane associated with $cr_2 = \dfrac{1}{cr_3}$ is presented. Both critical points have the same parameter plane, as the are conjugated; thus, only one of them is considered a free independent critical point. It is usual not to plot a parameter plane for $z = -1$ as, although it is also a critical point, it is a pre-image of a fixed point and this plane would give us information only about the stability of fixed point $z = 1$. We would like to remark that the outer black area corresponds to the stability of $z = 1$. Moreover, the right-sided black circle inside the red area corresponds to the values of the parameter where $f_5(\alpha)$ and $f_6(\alpha)$ are simultaneously attracting. In the detail in Figure 3b, the end of both inner antennas can be observed and also, in red color, the wide area of stable values of α, where only the convergence to the roots is possible. This parameter plane has been obtained by using a mesh of 500×500 points for complex values of α and a maximum of 200 iterations.

3.3. Dynamical Planes

If one of the values of α (being painted red or black in the parameter plane) is selected, an specific member of the class of iterative methods (5) is chosen. Then, a set of initial estimations can be used in order to observe the performance of this iterative method.

The dynamical plane obtained by iterating an element of the family under study, is obtained by using each point of a mesh of 400×400 points of the complex plane as an initial estimation. Those points with orbits converging to infinity appear in blue color; those that converge to zero are painted in orange (both with a precision of 10^{-3}) and in green, red, etc., those converge in to other fixed points. All the fixed points appear as white stars in the figures when they are attractors or as white circles when they are repulsors. Moreover, the point is painted in black if the maximum number of 80 iterations is reached without converging to any of the fixed points. The routines used are slight modifications of those appearing in [24].

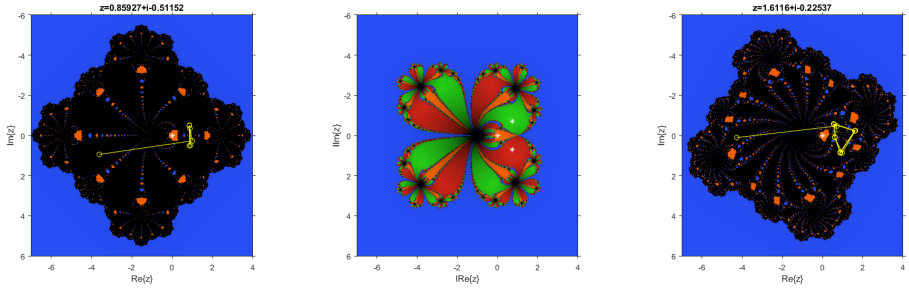
Thus, various stable elements can be chosen: values of α where no attracting periodic points nor strange fixed points appear. Some of them can be observed at the dynamical planes of Figure 4. In particular, Figure 4a, shows the performance of the method corresponding to $\alpha = 1$, where the only basins of attraction are those of $z = 0$ (orange area) and $z = \infty$ (blue region). Indeed, these basins are wider in this case than in cases $\alpha = -20i$ (Figure 4b) and $\alpha = 5 - 10i$ (Figure 4c).



(**a**) $\alpha = 1$ (**b**) $\alpha = -20i$ (**c**) $\alpha = 5 - 10i$

Figure 4. Stable performance of operator $O_\alpha(z)$.

Similarly, some unstable elements can be chosen. It can be observed that $\alpha = -50$, which yields in the biggest black area of parameter plane (see Figure 3), corresponds to an iterative method whose bigger basin of attraction is that of an attracting 2-periodic orbit, which can be observed in yellow in Figure 5a. However, Figure 5b, where $\alpha = 25$, the basins of attraction of attracting strange fixed points $f_5(\alpha)$ and $f_6(\alpha)$ appear in red and green, and are wider than the one of $z = 0$. Other attracting elements appear for different values of the parameter; for example, $\alpha = -20 + 45i$ gives rise to a dynamical plane (see Figure 5c) where the black basin corresponds to an attracting periodic orbit of period four, which has been plotted in yellow in the figure. Therefore, it can be observed that the fractal defined by the Julia set, the boundary among the basins of attraction, is much more complicated in the case of unstable elements of the family of iterative methods.



(**a**) $\alpha = -50$ (**b**) $\alpha = 25$ (**c**) $\alpha = -20 + 45i$

Figure 5. Unstable performance of operator $O_\alpha(z)$.

This information will be checked numerically in the next section, where non-polynomial functions are used to see the performance of some of these stable and unstable elements of the proposed family of iterative methods.

4. Numerical Results

In this section, we demonstrate the behaviour of the new family of iterative methods defined in (5), which is a particular sub-class of the method defined in (3). From the stability analysis, we know that those members of this class of iterative schemes, corresponding to values of α inside the red area of the parameter plane (see Figure 3), have better performance on quadratic polynomials than those belonging to black areas. We check now their performance on other kind of functions in order to see whether these stability properties are held.

Numerical computations have been carried out by using *MATLAB* R2019a, by using variable precision arithmetics with 1000 digits of mantissa, on a PC equipped with a *Intel* Core™ i5-5200U CPU 2.20GHz. In all the tables, we demonstrate the residuals $|f(x_{k+1})|$ and $|x_{k+1} - x_k|$ at the last iteration, the estimation of the solution found, the number of iterations needed (if the scheme does not converge, a "d" appears) and the execution time

in seconds with the format $t \pm d$, calculated with the command *cputime*. In it, t is the mean of 100 consecutive executions and d is the standard deviation. The stopping criterion used is $|x_{k+1} - x_k| + |f(x_{k+1})| < 10^{-200}$, and the ACOC, defined in [26], has been also presented in the tables, whose expression is:

$$ACOC = \frac{\ln|(x_{k+1} - x_k)/(x_k - x_{k-1})|}{\ln|(x_k - x_{k-1})/(x_{k-1} - x_{k-2})|}, \quad k = 2, 3, \ldots$$

The nonlinear test functions used are both academical and real-life functions:

- $f_1(x) = sin^2 x - x^2 + 1$, with two real roots at $x_1^* \approx -1.4044$ and $x_2^* \approx 1.4044$.
- $f_2(x) = cosx - xe^x$, with real roots at $x_1^* \approx 0.517$, $x_2^* \approx -14.137$ and $x_3^* \approx -17.278$, among others.
- Colebrook-White function [27] $f_3(x) = \sqrt{\frac{1}{f}} + 0.86 \ln\left(\frac{10^{-4}}{3.7} + \frac{2.51}{10^5\sqrt{\frac{1}{f}}}\right)$, with a real root at $x^* \approx 0.01885050$, among others.

Colebrook-White function is one of the most accurate equations for the calculation of the friction factor and over a wider range, but it has the disadvantage of complexity, as it is an implicit function. It must be solved iteratively until an acceptable level of error is reached, with the computational cost and time involved. It was proposed by Colebrook and White in 1939 [4,27]. and is the most widely used because it is the most accurate and universal. For each nonlinear function, two initial approximations are used: one near to the solution and the other far from it.

Some of the proposed schemes are compared with classical Newton' and Jarratt's methods, Wang. et al. scheme KM [28] and two procedures from Chun [29], denoted by CM1 and CM2.

In Tables 1 and 2, we notice that when values of α are considered inside the stable zone ($\alpha = 1$, $\alpha = -20i$, $\alpha = 5 - 10i$ or $\alpha = -4.5 + 10i$), the method achieves better approximations to any of the roots with fewer iterations. It is noteworthy that one of the unstable values of the parameter, $\alpha = -20 + 45i$, does not even obtain convergence. One of the stable members for $\alpha = 1$, shows the lowest execution time, even compared with the classical Newton and Jarratt's methods.

Table 1. $f_1(x) = sin^2 x - x^2 + 1$, $x_0 = 2$.

α	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	Solution	Iterations	ACOC	Time (s)
-50	7.0883×10^{-1007}	8.4884×10^{-253}	-1.4044	8	4.0	0.3223 ± 0.2001
-40	0	1.4357×10^{-402}	-1.4044	29	4.0	0.9086 ± 0.5325
$-20 + 45$i	d	d	d	d	d	d
$-16 - 45$i	2.4842×10^{-1432}	5.1013×10^{-425}	1.4044	10	4.0	0.3597 ± 0.1498
1	0	1.8974×10^{-331}	1.4044	6	4.0	$0.2188\ 0.2253 \pm 0.0838$
-20i	7.0302×10^{-1505}	8.8753×10^{-498}	1.4044	7	4.0	0.2487 ± 0.0415
$5 - 10$i	0.0	1.4936×10^{-755}	1.4044	7	4.0	$0.2481\ 0.2419 \pm 0.0176$
$-4.5 + 10$i	3.5457×10^{-891}	1.0501×10^{-223}	1.4044	6	4.0	0.2223 ± 0.0224
Newton	1.4479×10^{-514}	8.6274×10^{-258}	1.4044	10	2.0	0.2033 ± 0.0248
Jarratt	0.0	9.6997×10^{-510}	1.4044	6	4.0	0.1970 ± 0.0200
*CM*1	0.0	4.9393×10^{-810}	1.4044	5	6.0	0.2264 ± 0.0419
*CM*2	0.0	1.5533×10^{-693}	1.4044	5	6.0	0.2334 ± 0.0619
KM	0.0	9.821×10^{-761}	1.4044	5	6.0	0.2158 ± 0.0185

Table 2. $f_1(x) = sin^2x - x^2 + 1$, $x_0 = 4$.

α	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	Solution	Iterations	ACOC	Time (s)
-50	0	1.5957×10^{-569}	1.4044	7	4.0	0.2347 ± 0.0293
-40	0	3.5896×10^{-451}	1.4044	7	4.0	0.2420 ± 0.0614
$-20 + 45i$	1.1188×10^{-1416}	1.475×10^{-409}	1.4044	8	4.0	0.2589 ± 0.0262
$-16 - 45i$	2.9743×10^{-1380}	8.4532×10^{-372}	1.4044	8	4.0	0.2603 ± 0.0305
1	0	7.1697×10^{-507}	1.4044	7	4.0	0.2263 ± 0.0155
$-20i$	1.1363×10^{-995}	6.6845×10^{-250}	1.4044	7	4.0	0.2347 ± 0.0266
$5 - 10i$	3.0563×10^{-1262}	2.6576×10^{-316}	1.4044	7	4.0	0.2314 ± 0.0152
$-4.5 + 10i$	0	3.7277×10^{-384}	1.4044	7	4.0	0.2302 ± 0.0129
Newton	1.6796×10^{-543}	2.9384×10^{-272}	1.4044	11	2.0	0.2013 ± 0.0128
Jarratt	0.0	1.7167×10^{-754}	1.4044	7	4.0	0.1970 ± 0.0200
*CM*1	0.0	7.3559×10^{-390}	1.4044	5	6.0	0.2177 ± 0.0203
*CM*2	0.0	2.2786×10^{-230}	1.4044	5	6.0	0.2100 ± 0.0144
KM	0.0	9.2909×10^{-265}	1.4044	5	6.0	0.2083 ± 0.0436

In Tables 3 and 4, the best results are obtained by the stable proposed members of family (5), in terms of the number of iterations and residuals. Two of the unstable members do not reach convergence to any root.

Table 3. $f_2(x) = cosx - xe^x$, $x_0 = 1$.

α	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	Solution	Iterations	ACOC	Time (s)
-50	1.0082×10^{-1007}	4.6325×10^{-399}	-14.137	8	4.0	0.2728 ± 0.0657
-40	d	d	d	d	d	d
$-20 + 45i$	2.9787×10^{-1009}	2.2945×10^{-324}	-17.278	12	4.0	0.3858 ± 0.0756
$-16 - 45i$	d	d	d	d	d	d
1	0	5.7578×10^{-315}	0.517	6	4.0	0.2247 ± 0.0474
$-20i$	1.9236×10^{-1366}	6.3944×10^{-359}	0.517	7	4.0	0.2566 ± 0.0551
$5 - 10i$	0.0	6.7181×10^{-632}	0.517	7	4.0	0.2797 ± 0.0715
$-4.5 + 10i$	3.0281×10^{-1747}	1.7704×10^{-740}	0.517	7	4.0	0.3034 ± 0.0963
Newton	1.4521×10^{-498}	7.5503×10^{-250}	0.517	10	2.0	0.2075 ± 0.0499
Jarratt	0.0	3.9685×10^{-570}	0.517	6	4.0	0.2198 ± 0.0529
*CM*1	0.0	4.9393×10^{-810}	1.4044	5	6.0	0.1923 ± 0.0152
*CM*2	0.0	1.5533×10^{-693}	1.4044	5	6.0	0.1922 ± 0.0168
KM	0.0	9.821×10^{-761}	1.4044	5	6.0	0.2158 ± 0.0185

Table 4. $f_2(x) = cosx - xe^x$, $x_0 = 2.4$.

α	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	Solution	Iterations	ACOC	Time (s)
-50	1.4177×10^{-1008}	5.6457×10^{-294}	-1.8639	6	4.0	0.1848 ± 0.0133
-40	1.4177×10^{-1008}	5.1924×10^{-678}	-1.8639	7	4.0	0.2048 ± 0.0155
$-20 + 45i$	d	d	d	d	d	d
$-16 - 45i$	d	d	d	d	d	d
1	0	1.6078×10^{-702}	0.5177	8	4.0	0.2273 ± 0.0189
$-20i$	1.0214×10^{-961}	1.0043×10^{-240}	-1.8639	9	4.0	0.2553 ± 0.0230
$5 - 10i$	0.0	4.4053×10^{-627}	0.5177	22	4.0	0.5289 ± 0.0233
$-4.5 + 10i$	1.4234×10^{-979}	2.2838×10^{-242}	-29.8451	9	4.0	0.2653 ± 0.0487
Newton	2.4269×10^{-437}	3.0867×10^{-219}	0.5177	12	2.0	0.1956 ± 0.0260
Jarratt	0.0	1.0119×10^{-612}	0.5177	7	4.0	0.1909 ± 0.0243
*CM*1	0.0	4.8037×10^{-559}	0.5177	6	6.0	0.2230 ± 0.0273
*CM*2	0.0	1.5709×10^{-609}	0.5177	6	6.0	0.2208 ± 0.0261
KM	0.0	9.7594×10^{-581}	0.5177	5	6.0	0.2106 ± 0.0218

In the case of the Colebrook-White function, as shown in Tables 5 and 6, it is only possible to converge to the solution with a very close initial estimation. In this case, the

best results in terms of the number of iterations are obtained by the stable members of the proposed class as well as by Jarratt's method. In terms of the lowest execution time, the best results are given by Jarratt and $\alpha = 1$ schemes.

Table 5. $f_3(x) = \sqrt{\frac{1}{f}} + 0.86 \ln\left(\frac{10^{-4}}{3.7} + \frac{2.51}{10^5\sqrt{\frac{1}{f}}}\right)$, $x_0 = 0.02$.

α	$\lvert f(x_k)\rvert$	$\lvert x_{k+1} - x_k\rvert$	Solution	Iterations	ACOC	Time (s)
-50	2.2683×10^{-1007}	5.9163×10^{-668}	0.01885050	6	4.0	0.4098 ± 0.1367
-40	1.1341×10^{-1007}	3.1316×10^{-702}	0.01885050	6	4.0	0.3978 ± 0.0532
$-20 + 45$i	2.4166×10^{-667}	0.0	0.01885050	6	4.0	0.4517 ± 0.1145
$-16 - 45$i	1.5247×10^{-1677}	1.4073×10^{-671}	0.01885050	6	4.0	0.4555 ± 0.0999
1	1.1341×10^{-1007}	2.2713×10^{-277}	0.01885050	5	4.0	0.3361 ± 0.0419
-20i	4.0218×10^{-796}	9.0943×10^{-202}	0.01885050	5	4.0	0.3386 ± 0.0255
$5 - 10$i	1.5202×10^{-872}	8.1035×10^{-221}	0.01885050	5	4.0	0.3744 ± 0.0241
$-4.5 + 10$i	3.2812×10^{-893}	5.7307×10^{-226}	0.01885050	5	4.0	0.3706 ± 0.0178
Newton	2.9268×10^{-693}	5.9571×10^{-349}	0.01885050	9	2.0	0.3630 ± 0.0332
Jarratt	1.1341×10^{-1007}	2.0266×10^{-489}	0.01885050	5	4.0	0.3319 ± 0.0655
*CM*1	1.1341×10^{-1007}	2.0266×10^{-489}	0.01885050	5	4.0	0.3447 ± 0.0301
*CM*2	1.1341×10^{-1007}	2.0266×10^{-489}	0.01885050	5	4.0	0.3494 ± 0.0252
KM	1.1341×10^{-1007}	2.0266×10^{-489}	0.01885050	5	4.0	0.3400 ± 0.0208

Table 6. $f_3(x) = \sqrt{\frac{1}{f}} + 0.86 \ln\left(\frac{10^{-4}}{3.7} + \frac{2.51}{10^5\sqrt{\frac{1}{f}}}\right)$, $x_0 = 0.009$.

α	$\lvert f(x_k)\rvert$	$\lvert x_{k+1} - x_k\rvert$	Solution	Iterations	ACOC	Time (s)
-50	d	d	d	d	d	d
-40	d	d	d	d	d	d
$-20 + 45$i	d	d	d	d	d	d
$-16 - 45$i	d	d	d	d	d	d
1	2.3085×10^{-906}	4.1347×10^{-229}	0.01885050	6	4.0	0.3880 ± 0.0206
-20i	2.2683×10^{-1007}	1.9203×10^{-445}	0.01885050	9	4.0	0.6564 ± 0.0519
$5 - 10$i	1.2533×10^{-1118}	2.48×10^{-282}	0.01885050	7	4.0	0.5059 ± 0.0328
$-4.5 + 10$i	2.2683×10^{-1007}	2.5742×10^{-397}	0.01885050	7	4.0	0.5020 ± 0.0177
Newton	2.0584×10^{-397}	4.9958×10^{-201}	0.01885050	10	2.0	0.3942 ± 0.0227
Jarratt	1.1341×10^{-1007}	1.1036×10^{-272}	0.01885050	5	4.0	0.3200 ± 0.0233
*CM*1	0	1.8787×10^{-675}	0.01885050	5	6.0	0.4191 ± 0.0255
*CM*2	0	1.8799×10^{-675}	0.01885050	5	6.0	0.4194 ± 0.0354
KM	0	1.8793×10^{-675}	0.01885050	5	6.0	0.4064 ± 0.0168

5. Conclusions

In this manuscript, we present a new parametric family of iterative methods with the fourth-order of convergence in order to solve nonlinear equations $f(x) = 0$. The class has been generated using composition and weight functions techniques. With the help of dynamical analysis, we select the most stable methods by choosing some values of the free parameter. These schemes demonstrate a simple fractal as the Julia set separating the basins of attraction. The described numerical examples allow us to confirm the theoretical results corresponding to the proposed convergence and stability. On the other hand, when the parameter alpha is unstable, the corresponding method needs more iterations to converge or fails to converge. These numerical tests confirm that the new family of methods is suitable for solving nonlinear problems, when the adequate values of the free disposable parameter are used.

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3.2. Articulo 2

En este artículo presentamos una modificación de la familia paramétrica previa para resolver ecuaciones no lineales. En esta nueva clase, adaptada al cálculo de raíces múltiples, se demuestran las condiciones que le permiten converger con orden cuatro, con lo que los elementos de la misma son esquemas iterativos óptimos. Además se lleva a cabo un análisis dinámico completo, que permite seleccionar aquellos miembros con mejores propiedades de estabilidad y se compara su desempeño sobre diferentes problemas no lineales, tanto académicos como de la vida real, con los proporcionados por esquemas ya existentes. Una presentación oral de algunos resultados parciales de este artículo se expuso en la en la XXIV edición del congreso Modelling for Engineering & Human Behaviour 2022 (Valencia).

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Parametric Iterative Method for Addressing an Embedded-Steel Constitutive Model with Multiple Roots

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Abstract: In this paper, an iterative procedure to find the solution of a nonlinear constitutive model for embedded steel reinforcement is introduced. The model presents different multiplicities, where parameters are randomly selected within a solvability region. To achieve this, a class of multipoint fixed-point iterative schemes for single roots is modified to find multiple roots, achieving the fourth order of convergence. Complex discrete dynamics techniques are employed to select the members with the most stable performance. The mechanical problem referred to earlier, as well as some academic problems involving multiple roots, are solved numerically to verify the theoretical analysis, robustness, and applicability of the proposed scheme.

Keywords: nonlinear constitutive models; iterative methods; stability; order of convergence; unified parameter plane; multiple roots

MSC: 65H05; 37F10

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1. Introduction

Nonlinearity usually arises in structural models [1,2]. In particular, materials such as reinforced concrete involve nonlinear stress–strain relationships in the mechanical model. The application of these model implementations usually calls for the application of numerical approaches to find solutions of the related nonlinear equations $f(x) = 0$, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$, such as Newton-type methods [3]. The most appropriate algorithm for solving a nonlinear problem is usually a balance between computational cost and precision [4]. Nevertheless, in spite of the highly nonlinear character of the problem, it is, however, possible to increase the efficiency of the method by previously defining a solvability domain, which is derived using algebraic procedures.

Newton's method cannot be applied at a point where the first derivative cancels out. Graphically, it means that the tangent line to the curve $f(x)$ at this point is horizontal, and therefore it does not intersect the x-axis.

Let x^* be a multiple root of $f(x) = 0$ with multiplicity m. Then, it is also a multiple root of $f'(x) = 0$ of multiplicity $m - 1$, of $f''(x) = 0$ with multiplicity $m - 2$, and so on [5]. The fact that the function does not change sign precludes the use of methods that use intervals. At multiple roots, not only $f(x)$ but also $f'(x)$ approaches zero, which affects Newton's method since its denominator includes the first derivative. In general, iterative procedures to solve nonlinear equations with single roots reduce their order of convergence when the equations have multiple roots ($m > 1$), or even diverge.

Many authors have proposed modifications to Newton's method for multiple roots. Two of the best-known are Rall's method [6], when we know the multiplicity,

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots \tag{1}$$

and Schröder's method [7],

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{[f'(x_k)]^2 - f(x_k)f''(x_k)}, \quad k = 0, 1, 2, \ldots, \tag{2}$$

when the multiplicity is unknown. In recent years, some optimal iterative methods for solving nonlinear problems with multiple roots have appeared in the literature, such as [8–11], for cases with known multiplicity and [12], for unknown multiplicity. In most cases, the iterative expression is complicated, which increases the computational cost.

So, our aim is to design a fourth-order optimal iterative method able to find multiple roots with known multiplicity but with low computational cost. Based on the previous study [13] of the multipoint fixed-point class of fourth-order iterative schemes

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - H(\bar{\eta})\frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{3}$$

where $H(\bar{\eta})$ is a real function of variable $\bar{\eta} = \dfrac{f'(y_k)}{f'(x_k)}$ that satisfies $H'''(1)$ is bounded, and $H''(1) = \frac{9}{4}$, $H'(1) = -\frac{3}{4}$, and $H(1) = 1$. We propose the iterative scheme for solving nonlinear equations with $m > 1$

$$\begin{aligned} y_k &= x_k - a\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - G(\eta)\frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{4}$$

where a is a real parameter, and $G(\eta)$ is the weight function of variable $\eta = \left(\dfrac{f'(x_k)}{f'(y_k)}\right)^{\frac{1}{m-1}}$.

This manuscript is organized as follows. Section 2 introduces the steel constitutive model and its solvability region. In Section 3, we design a class of iterative methods for multiple roots in order to solve the problem described in Section 2 and we analyze the convergence. In Section 4, we perform a stability analysis of the described family, finding the most stable members. Finally, Section 5 is devoted to the applicability of the stable members of the new family for solving the steel constitutive model and some academic problems. We also compare the performance of the introduced schemes with the classical Rall's method, which also needs to know the multiplicity.

2. The Embedded-Steel Constitutive Model and Its Solvavility

Various authors [14–16] regard steel reinforcement stiffened by the concrete adhered to it as the "embedded bar model". One of these approaches, the Refined Compression Field Theory (RCFT) [17], involves an equilibrium constraint in the steel model that accounts for the tensile stiffening action of the concrete between cracks. As a consequence, a nonlinear equality is entered into the steel constitutive equation in terms of the apparent yield strain. The latter theory would predict the average stress of an embedded bar as a dependence on

the average strain (i.e., as measured over a given length that included a variety of cracks) in the following way:

$$\sigma_{s,av} = \begin{cases} f_y - \dfrac{A_c}{A_s}\dfrac{f_{ct}}{1+\sqrt{3.6M\varepsilon_{s,av}}}, & \varepsilon_{s,av} \geq \varepsilon_{max}, \\ E_s\varepsilon_{s,av}, & \varepsilon_{s,av} < \varepsilon_{max}, \end{cases} \tag{5}$$

with

$$\varepsilon_{max} = \frac{f_y}{E_s} - \frac{f_{ct}A_c}{E_sA_s\left(1+\sqrt{3.6M\varepsilon_{max}}\right)}, \quad M = \frac{A_c}{\pi\phi} \tag{6}$$

where E_s is the elastic modulus of steel; f_y is the yield point of steel; f_{ct} is the tensile strength of concrete; $\sigma_{s,av}$ and $\sigma_{ct,av}$ are the average tensile stresses in the stiffened steel and in the concrete, respectively; A_s is the cross-section of the steel bars; $\epsilon_{s,av}$ is the average strain in the reinforcing bar; A_c is the transverse area of concrete adhered to the bar, which participates in the tensile stiffening action; and M is a parameter dependent on the bond characteristics between concrete and steel. The previous statement is based on the stress balance between a cracked section, where only reinforcement is contributing, and a general section, where both reinforcement and the surrounding concrete are participating. Figure 1 represents the schematic.

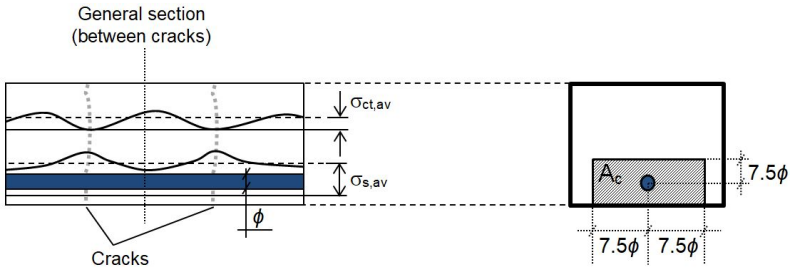


Figure 1. Profiles of average stresses ($\sigma_{ct,av}$ and $\sigma_{st,av}$) and area of prescribed tensile stiffening (A_c) actually attached to the steel.

Technical codes [18] suggest a value for A_c that is equal to the area surrounding the bar at a distance no greater than 7.5 ϕ from the center of the bar, where ϕ is the diameter of the bar. It was found that for certain specimens (namely those with high values of the ratio f_{ct}/ρ, where ρ is the reinforcing ratio), when this value is taken, it is not feasible to obtain a real positive solution for the effective yield stress ϵ_{max} defined in (5). Furthermore, if the solvability analysis is stated in accordance with the variables f_{ct} and ϵ_{max}, and the area value A_c is increased monotonically, beyond a given value of this area, the bijection is broken between f_{ct} and ϵ_{max} (uniqueness problem). Later, for higher area values of A_c up to the prescribed values provided by the technical codes, one encounters the lack of a real solution (existence problem) [19]. Thus, the internal stress equilibrium along the cracked element is not found.

In [19], the greatest value of the area A_c, in which the constitutive model of the embedded steel has at least one real positive solution, is determined. In this regard, the greatest part of the tensile stiffening area that can be considered to maintain the solvability of the constitutive model (that is, to preserve the internal force equilibrium, so that by increasing the concrete involvement, the steel tension is reduced) is expressed by the coefficient

$$\lambda = \frac{A_sf_y}{A_cf_{ct}}\left(\frac{2}{3} + \frac{\sqrt{\left(1+10.8M\epsilon_y\right)^3}}{48.6M\epsilon_y}\right) \tag{7}$$

where $\epsilon_y = f_y/E_s$ is considered to be the deflection corresponding to the yield strength of the steel. Thus, the coefficient λ is the boundary of the solvability domain for the constitutive model of the embedded steel suggested by the RCFT. As this region is only

derived through algebraic calculations, it can be applicable to any experimentally based approach to the tensile stiffening model of concrete.

For some design cases, the above limit may be below the value required by the technical codes for the area A_c; therefore, the region of resolubility is well within the design range required by the technical codes. In fact, various studies [17,20,21] indicate the appropriateness of correcting the tensile stiffening area to correct the shear performance of reinforced concrete elements, especially for high shear deformations, where technical codes underestimate the tensile stiffness of the concrete.

3. Convergence Analysis of the Proposed Class

In this section, the convergence analysis of Family (4) is performed.

Theorem 1. *Let us consider $f : I \subseteq \mathbb{R} \to \mathbb{R}$ as a differentiable enough function in an open interval I and let $x^* \in I$ be a multiple root of $f(x) = 0$, with multiplicity $m > 1$. Let also $G(\eta)$ be a real function satisfying $G(\mu) = m$, $G'(\mu) = \dfrac{-(a-m)^2(-1+m)m}{a(a-2m+am)}$, $G''(\mu) = \dfrac{m^4(2-2m-m^2+m^3)}{4(2+m)}$, and $|G'''(\mu)| < +\infty$, being $\mu = \dfrac{m}{m-a}$. If $a = \dfrac{2m}{2+m}$ and x_0 is an initial guess close enough to x^*, then the class of schemes (4) satisfies the following error equation,*

$$e_{k+1} = \left(\frac{c_1^3}{3(m-1)^3 m^{10}(m+2)^2} \left(m^5\left(m^9+3m^8-m^7-11m^6+6m^5+8m^4-42m^3+ 36m^2+48m-48\right)-32G'''(\mu)(m+2)^2\right) - \frac{c_2 c_1}{m} + \frac{c_3 m}{(m+2)^2} \right) e_k^4 + \mathcal{O}(e_k^5).$$

showing the fourth-order of convergence. Here, $\eta = \left(\dfrac{f'(x_k)}{f'(y_k)}\right)^{\frac{1}{m-1}}$, $c_k = \dfrac{m!}{(m+1)!}\dfrac{f^{(m+k)}(x^*)}{f^{(m)}(x^*)}$, $k = 2, 3, \ldots$, and $e_k = x_k - x^*$.

Proof. Let x^* be a multiple zero of f with multiplicity m. By hypothesis, f can be expanded using the Taylor series for $f(x_k)$ and $f'(x_k)$ about x^*,

$$\begin{aligned} f(x_k) &= \frac{f^{(m)}(x^*)}{m!} e_k^m (1 + e_k c_1 + e_k^2 c_2 + e_k^3 c_3 + e_k^4 c_4) + \mathcal{O}(e_k^5), \\ f'(x_k) &= \frac{f^{(m)}(x^*)}{m!} e_k^{m-1} \left(m + (m+1) e_k c_1 + (m+2) e_k^2 c_2 + (m+3) e_k^3 c_3\right) + \mathcal{O}(e_k^4). \end{aligned}$$

From these expressions, we obtain the expansion of the first step

$$\begin{aligned} y_k - x^* = {} & \left(1 - \frac{a}{m}\right) e_k + \frac{a c_1}{m^2} e_k^2 + \frac{-a(1+m)c_1^2 + 2amc_2}{m^3} e_k^3 + \\ & \frac{a}{m^4}\left((1+m)^2 c_1^3 - m(4+3m) c_1 c_2 + 3m^2 c_3\right) e_k^4 + \mathcal{O}(e_k^5). \end{aligned}$$

Expanding in Taylor's series $f'(y_k)$ around x^*,

$$\begin{aligned} f'(y_k) = {} & \frac{f^{(m)}(x^*)}{m!} y_k^{m-1} \Big(m + (m+1)(y_k - x^*) c_1 + (m+2)(y_k - x^*)^2 c_2 + \\ & (m+3)(y_k - x^*)^3 c_3\Big) + \mathcal{O}(e_k^4), \end{aligned}$$

and then,

$$\begin{aligned}\eta &= \frac{m}{m-a} - \frac{a(a-2m+am)c_1}{(a-m)^2(-1+m)m}e_k - \\ &\frac{a}{2(a-m)^3(m-1)^2m^2}\big(c_1^2\big(a^3(m+1)^2 - 2a^2(m+1)^2(2m-1) + \\ &am\big(3m^3+12m^2-5m-6\big) - 6m^2\big(m^2-1\big)\big) - \\ &2c_2(m-1)\big(a^3(m+2) - 4a^2m(m+2) + 3am^2(m+4) - 6m^3\big)\big)e_k^2 - \\ &\frac{a}{6(m-1)^3m^4(a-m)^4}\Big(6c_3(m-1)^2m(a-m)^2\big(a^3(m+3) - 4a^2m(m+3) + \\ &6am^2(m+3) - 12m^3\big) - 6c_2c_1(m-1)m\big(a^5\big(m^2+3m+2\big) - \\ &2a^4\big(3m^3+9m^2+5m-2\big) + a^3m\big(15m^3+43m^2+16m-20\big) - \\ &2a^2m^2\big(8m^3+30m^2+3m-20\big) + 2am^3\big(3m^3+22m^2+m-20\big) - \\ &4m^4\big(3m^2+m-4\big)\big) + c_1^3\big(a^5(m+1)^3(2m-1) - 6a^4m(m+1)^2\big(2m^2+m-2\big) + \\ &3a^3m(m+1)^2\big(10m^3+3m^2-13m+2\big) - \\ &2a^2m^2\big(16m^5+52m^4-21m^3-50m^2-5m+12\big) + \\ &12am^3\big(m^5+7m^4-3m^3-10m^2+2m+3\big) - 24m^4\big(m^2-1\big)^2\big)\Big)e_k^3 + \mathcal{O}(e_k^4),\end{aligned}$$

that is, η tends to $\mu = \frac{m}{m-a}$ when $k \to \infty$. We expand $G(\eta)$ about $\mu = \frac{m}{m-a}$,

$$G(\eta) \approx G(\mu) + G'(\mu)(\eta-\mu) + \frac{1}{2}G''(\mu)(\eta-\mu)^2 + \frac{1}{6}G'''(\mu)(\eta-\mu)^3,$$

and the error equations yields

$$\begin{aligned}e_{k+1} &= \left(1 - \frac{G(\mu)}{m}\right)e_k + \left(G(\mu) + \frac{aG'(\mu)(a-2m+am)}{(a-m)^2(-1+m)}\right)\frac{c_1}{m^2}e_k^2 - \\ &\frac{1}{2(m-1)^2m^3(a-m)^4}\Big(-a^2c_1^2G''(\mu)(am+a-2m)^2 \\ &+a\Big(G'(\mu)(a-m)\Big(c_1^2\Big(a^3(m+1)^2 - 2a^2(m+1)^2(2m-1) \\ &+am\Big(3m^3+12m^2-5m-6\Big) - 6m^2\Big(m^2-1\Big)\Big) \\ &-2c_2(m-1)\Big(a^3(m+2) - 4a^2m(m+2) + 3am^2(m+4) - 6m^3\Big)\Big) \\ &-ac_1^2G''(\mu)(am+a-2m)^2\Big) + aG'(\mu)(a-m)\Big(c_1^2\Big(a^3(m+1)^2 \\ &-2a^2(m+1)^2(2m-1) + am\Big(3m^3+12m^2-5m-6\Big) - 6m^2\Big(m^2-1\Big)\Big) \\ &-2c_2(m-1)\Big(a^3(m+2) - 3am^2(m+4) - 6m^3\Big)\Big) \\ &-2G(\mu)(m-1)^2(a-m)^4\Big(c_1^2(m+1) - 2c_2m\Big) \\ &-4ac_1^2G'(\mu)(m-1)(am+a-2m)(a-m)^2\Big)e_k^3 + \mathcal{O}(e_k^4).\end{aligned}$$

To make the coefficients of e_k and e_k^2 null (and achieve third-order convergence), the values of $G(\mu)$ and $G'(\mu)$ must satisfy

$$G(\mu) = m, \quad G'(\mu) = -\frac{(a-m)^2(-1+m)m}{a(a-2m+am)}.$$

In this case, the error equation depends on a and $G''(\mu)$:

$$\begin{aligned} e_{k+1} = \frac{1}{2m^3}\Bigg(c_1\Bigg(-2m^2 - \frac{a^2G''(\mu)(a-2m+am)^2}{(m-1)^2(a-m)^4} + \\ \frac{m}{(m-1)(a-m)(a-2m+am)}\Big(\Big(6a-2a^3\Big)m - a^3 + \Big(4a^2-12a\Big)m^3 + 2a^2 + \\ \Big(-a^3+6a^2+5a-6\Big)m^2 + (6-3a)m^4\Big)\Big) + c_2\Bigg(\frac{1}{a-2m+am}\Big(-2\Big(-a^2+6a-2\Big)m^2 + \\ 4a^2m - 2(3a-6)m^3\Big)\Big)\Bigg)e_k^3 + Q_4e_k^4 + \mathcal{O}(e_k^5), \end{aligned}$$

where $Q_4 = -\frac{1}{6(m-1)^3m^4(m-a)^6(am+a-2m)}(6c_3(m-1)^3m(a^2(m+3)3am(m+3)+6m^2)(a-m)^7 - 6c_1c_2(m-1)(a-m)^2(a^7(m-1)m(m+1)(m+2) + a^6(m+2)(G''(\mu)(m+1)^2 - 8m^4 - m^3 + 12m^2 - 3m) + a^5m((m-1)(25m^4+87m^3+48m^2-40m) - G''(\mu)(m+1)(m+2)(3m+7)) + 2a^4m^2(G''(\mu)(9m^2+26m+19) - (m-1)m(2m+3)(m(10m+27)-19)) + a^3m^3((m-1)(35m^4+193m^3+102m^2-180m) - 12G''(\mu)(3m+4)) + a^2m^4(24G''(\mu) - (m-1)m(16m^3+129m^2+87m-166)) + 3am^6(m^4+14m^3+m^2-44m+28) - 6(m-1)^2m^7(m+3)) + c_1^3(a^9(m-1)(m+1)^3(2m-1) + a^8(m+1)^2(3G''(\mu)(m+1)^2 + 7m^3 + 10m^2+17m-24) - a^7(m+1)^2(3G''(\mu)(m(5m+8)-3)(m+1) + G'''(\mu)(m+1)^2 - 84m^5 + 6m^4 + 210m^3 - 144m^2 + 12m) + a^6m(m+1)^2(3G''(\mu)(7m^3+40m^2+17m-24) + 8G'''(\mu)(m+1) - (m-1)m(196m^3+275m^2-497m+96)) - a^5m^2(3G''(\mu)(3m^4+50m^3+118m^2 - 14m - 77)(m+1) + 24G'''(\mu)(m+1)^2 - 2(m-1)(140m^6+581m^5+198m^4-691m^3 - 256m^2+168m)) + a^4m^3(6G''(\mu)(m(m(m(9m+65)+77)-49)-62) + 32G'''(\mu)(m+1) - 3(m-1)m(84m^5+445m^4+169m^3-671m^2-193m+222)) + 2a^3m^4(-6G''(\mu)(m+1)(-25 + 18m + 9m^2) - 8G'''(\mu) + (m-1)m(70m^5+499m^4+264m^3-959m^2-248m+402)) + a^2(m-1)m^5(24G''(\mu)(3m+4) - m(44m^5+461m^4+387m^3-1141m^2-331m+588) + 3a(m-1)^2m^7(2m^4+41m^3+96m^2-31m-80) - 6(m-1)^3m^8(m+1)(2m+7))).$

Solving the following system to achieve fourth-order convergence,

$$\left.\begin{aligned} -2m^2 - \frac{a^2G''(\mu)(a-2m+am)^2}{(m-1)^2(a-m)^4} + \\ \frac{m}{(m-1)(a-m)(a-2m+am)}\big(\big(6a-2a^3\big)m - a^3 + \big(4a^2-12a\big)m^3 + \\ 2a^2 + \big(-a^3+6a^2+5a-6\big)m^2 + (6-3a)m^4\big) &= 0, \\ -2\big(-a^2+6a-2\big)m^2 + 4a^2m - 2(3a-6)m^3 &= 0, \\ -4m^2 - \frac{2m(-6m^3+a^3(2+m)-4a^2m(2+m)+3am^2(4+m))}{(a-m)(a-2m+am)} &= 0, \end{aligned}\right\}$$

results in

$$G''(\mu) = \frac{m^4(2-2m-m^2+m^3)}{4(2+m)}, \quad a = \frac{2m}{2+m}.$$

Therefore, the error equation of (4) is

$$\begin{aligned} e_{k+1} = \Big(\frac{c_1^3}{3(m-1)^3m^{10}(m+2)^2}\big(m^5\big(m^9+3m^8-m^7-11m^6+6m^5+8m^4-42m^3 + \\ 36m^2+48m-48\big) - 32G'''(\mu)(m+2)^2\big) - \frac{c_2c_1}{m} + \frac{c_3m}{(m+2)^2}\Big)e_k^4 + \mathcal{O}(e_k^5). \end{aligned}$$

□

4. Stability Analysis

We have demonstrated that all the schemes of the proposed family converge with fourth-order accuracy. Now, we would like to determine which of them are less dependent on the initial estimation. This analysis is conducted in this section using tools of complex discrete dynamics on low-degree polynomials with multiple roots.

4.1. Basic Dynamical Concepts

In this section, the dynamical performance of the iterative schemes described in (4) is analyzed. First, some concepts are recalled [22,23]. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function, where $\hat{\mathbb{C}}$ is the Riemann sphere. The orbit of a point $z_0 \in \hat{\mathbb{C}}$ is defined as the successive application of the operator R on that point, determined by the set $\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}$, where $R^n(z_0)$ refers to applying n times R to z_0. In this case, R is calculated by applying the family of iterative schemes to $p(z)$, a low-degree polynomial.

A fixed point $z^F \in \hat{\mathbb{C}}$ of R is kept invariant after the application of the operator, satisfying $R(z^F) = z^F$. It should be noted that although all roots of the quadratic polynomial are fixed points of the operator R, we can find fixed points of R that are not roots of $p(z)$; these points are called strange fixed points. Any fixed point is classified as:

- Attracting, if $|R'(z^F)| < 1$,
- Superattracting, if $R'(z^F) = 0$,
- Repelling, if $|R'(z^F)| > 1$,
- Parabolic or neutral, if $|R'(z^F)| = 1$.

In this context, $R'(z^F)$ is called the stability function of fixed points z^F.

On the other hand, the attractor basins [24] define the final status of the orbit of any point in the complex plane after repeated application of the R operator. The basins of attraction of an attractive fixed point $z^F \in \hat{\mathbb{C}}$ are then defined as the collection of pre-images of any order satisfying

$$\mathcal{A}(z^F) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z^F, n \to +\infty\}.$$

Moreover, a point z_c is called a critical point of R if $R'(z_c) = 0$. The asymptotic behavior of the critical points is a key fact in analyzing the stability of the method. Previous results [25] state that at least one critical point appears in each immediate basin of attraction, that is, in the connected component of the basin of attraction containing the attractor. Let us also remark that superattracting fixed points are indeed critical points. Moreover, those critical points that are not roots of $p(z)$ are called free critical points.

The Fatou set of the rational function R, $\mathcal{F}(R)$, is the set of points $z \in \widehat{\mathbb{C}}$ whose orbits are tending to an attractor. Its complementary set in $z \in \widehat{\mathbb{C}}$ is the Julia set, $\mathcal{J}(R)$. So, the basin of attraction of any fixed point pertains to the Fatou set and the boundaries of these basins of attraction pertain to the Julia set.

Henceforth, the dynamical behavior of fourth-order parametric family (4) on the cubic polynomial $p(z) = (z-\alpha)^2(z-\beta)$, where $\alpha, \beta \in \mathbb{C}$, is analyzed. The weight function is

$$\begin{aligned} G(\eta) \quad = \quad & m - \frac{(a-m)^2(-1+m)m}{a(a-2m+am)}(\eta-\mu) + \\ & \frac{m^4(2-2m-m^2+m^3)}{8(2+m)}(\eta-\mu)^2 + \frac{G'''(\mu)}{6}(\eta-\mu)^3, \end{aligned} \tag{8}$$

where $a = \dfrac{2m}{2+m}$ and $\mu = \dfrac{m}{m-a}$.

A rational function $O_p(z)$ is obtained, depending on the parameter $G'''(\mu)$ of Class (4), named from now on as $G3$, and also on the roots α and β. To obtain a simpler operator, we use the conjugacy map [26] given by the Möbius transformation

$$M(z) = \frac{z-\alpha}{z-\beta}, \quad M^{-1}(z) = \frac{z\beta-\alpha}{z-1},$$

that satisfies

$$M(\infty) = 1, \quad M(\alpha) = 0, \quad M(\beta) = \infty,$$

yielding a rational function that no longer depends on α and β, and it is conjugated to $O_{G_3}(z)$ (and therefore, with equivalent dynamical behavior)

$$O_{G3}(z) = \left(M \circ O_p \circ M^{-1}\right)(z) = z^4 \frac{P(z,G3)}{Q(z,G3)},$$

where $P(z,G3) = (312 - 8G3 + (1014 - 24G3)z + (1368 - 12G3)z^2 + (1050 + 16G3)z^3 + (546 + 6G3)z^4 + (198 - 6G3)z^5 + (42 + G3)z^6 + 6z^7)$ and $Q(z,G3) = 768 + 2496z + 2928z^2 + 1140z^3 - (726 + 8G3)z^4 - (1176 + 24G3)z^5 - (678 + 12G3)z^6 - (204 - 16G3)z^7 - (24 - 6G3)z^8 + (12 - 6G3)z^9 + G3z^{10}$.

Solving the $O_{G3}(z) = z$ equation yields the fixed points of the rational function: $z = 0$ and $z = \infty$. These arise from the roots of the polynomial prior to the map of Möbius. The asymptotic behavior of all fixed points plays a key role in the stability of the iterative methods involved since their convergence to fixed points other than the roots is a major disadvantage for an iterative approach.

A straightforward result of the Möbius map applied to this rational function is the inverse conjugation,

$$\frac{1}{O_{G3}(z)} = O_{G3}\left(\frac{1}{z}\right).$$

Then:

- If z^F is a fixed point of $O_{G3}(z)$, that is, $O_{G3}(z^F) = z^F$, its conjugate $1/z^F$ is also a fixed point: $O_{G3}(1/z^F) = 1/z^F$.
- $z = 1$ is always a strange fixed point of $O_{G3}(z)$, arising from the divergence of the original operator, except maybe for some specific values of the parameters that simplify the operator.
- Given two conjugate fixed points, both have the same character since their stability function coincides as

$$O'_{G3}(z^F) = 1/O'_{G3}(z^F).$$

4.2. Performance of Strange Fixed Points

The stability of strange fixed points depends on $G3$. Now, the stability of $z = 1$ is stated (Figure 2).

Theorem 2. *The stability of the strange fixed point $z = 1$ is:*

- *Attracting, if $|G3 - 168| > 1152$,*
- *Repelling, if $|G3 - 168| < 1152$,*
- *Parabolic, if $|G3 - 168| = 1152$,*
- *Not a fixed point, if $G3 = 168$.*

Proof. $z = 1$ is a strange fixed point of $O_{G3}(z)$ if $G3 \neq 168$. Fixed point $z = 1$ is attracting if

$$|O'_{G3}(1)| = \left|\frac{-1152}{-168 + G3}\right| < 1 \leftrightarrow 1152 < |-168 + G3|.$$

Finally, if $G3$ satisfies $|G3 - 168| < 1152$, then $O'_{G3}(1) > 1$, and $z = 1$ is a repelling point. It is parabolic in $O'_{G3}(1) = 1$. □

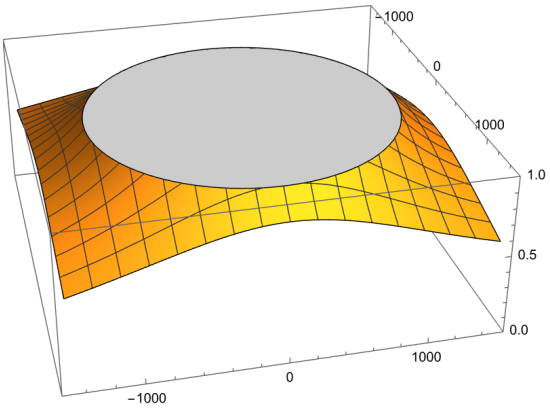


Figure 2. Stability region of $z = 1$.

Theorem 3. *The roots of* $T(x) = -768 - 3264x - 6192x^2 + (-7020 - 8G3)x^3 + (-5280 - 24G3)x^4 + (-2736 - 12G3)x^5 + (-1008 + 16G3)x^6 + (-258 + 6G3)x^7 + (-36 - 6G3)x^8 + (-6G3)x^9$ *are strange fixed points of* $O_{G3}(z)$*, unlike* $z = 1$*, and are denoted by* $f_i(G3)$*, i = 1, 2, . . . , 9. Moreover, four of them are superattracting for some values of G3 (and attracting around these values of G3):*

- $f_1(G3)$ *is superattracting for values of* $G3 \approx \{1.49735, 15.9586, 1242.25\}$,
- $f_2(G3)$ *is superattracting for values of* $G3 \approx 1.10566 \pm 0.31755i$,
- $f_4(G3)$ *is superattracting for values of* $G3 \approx 1.11091 \pm 0.314158i$,
- $f_7(G3)$ *is superattracting for values of* $G3 \approx -576.219 \pm 61.5856i$ *and* $1.10884 \pm 0.315802i$,
- $f_3(G3)$, $f_5(G3)$, $f_6(G3)$, $f_8(G3)$, *and* $f_9(G3)$ *are repelling, regardless of the value of parameter G3.*

In Figure 3, the stability function of $O_{G3}(z)$ can be observed. Let us remark that complex values of $G3$ inside region $|G3 - 168| < 1152$ define fourth-order iterative schemes whose numerical performance does not include divergence.

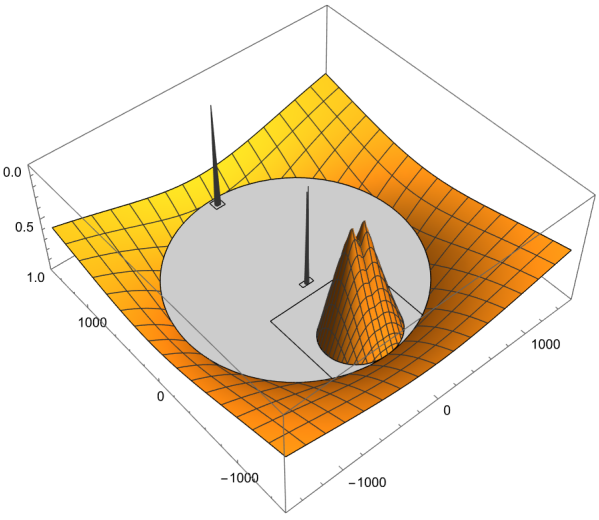


Figure 3. Regions of stability corresponding to $f_i(G3)$, $i = 1, 2, \ldots, 11$.

In addition, the existence of free critical points is essential for the stability of the family of iterative methods.

4.3. Critical Points and Parameter Planes

To calculate the critical points, we obtain

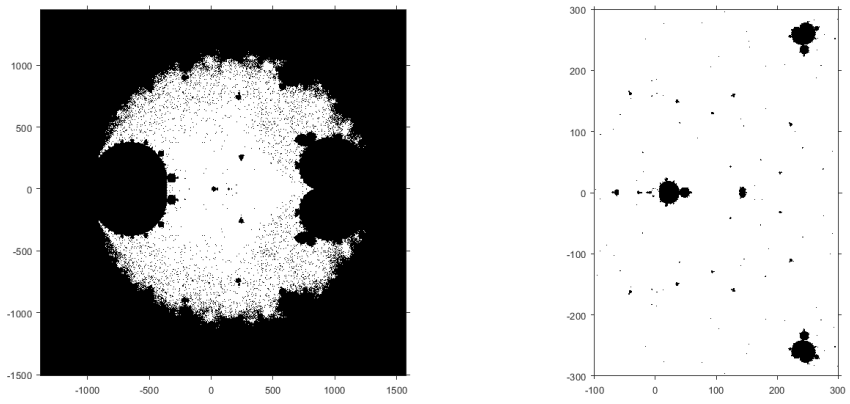
$$O'_{G3}(z) = \frac{R(z, G3)}{S(z, G3)},$$

where $R(z, G3) = 6z^3(4 + 5z + 2z^2 + z^3)^2(G3(-2 - 2z + z^2)^2(-64 - 108z + 80z^2 + 94z^3 - 46z^4 - 53z^5 - 12z^6 + z^7) + 6(1664 + 6656z + 10788z^2 + 8583z^3 + 2748z^4 - 876z^5 - 1404z^6 - 741z^7 - 192z^8 - 14z^9 + 4z^{10}))$ and $S(z, G3) = (768 + 2496z + 2928z^2 + 1140z^3 - 2(363 + 4G3)z^4 - 24(49 + G3)z^5 - 6(113 + 2G3)z^6 + 4(-51 + 4G3)z^7 + 6(-4 + G3)z^8 - 6(-2 + G3)z^9 + G3z^{10})^2$.

Since the order of the convergence of our iterative class of methods is greater than two, those fixed points from the original roots of $p(z)$, i.e., $z = 0$ and $z = \infty$, are also both critical points. In the next result, the remaining critical points, called free critical points, are also determined.

Theorem 4. *The set of free critical points cr_i, $i = 1, 2, \ldots, 13$, of fixed point operator $O_{G3}(z)$ is composed of $cr_1 = -1$, $cr_2 = \frac{1}{2}(-1 + i\sqrt{15})$, and $cr_3 = \frac{1}{2}(-1 + i\sqrt{15})$, which are pre-images of the strange fixed point $z = 1$ and the roots of the polynomial $t(x) = 9984 - 256G3 + (39936 - 944G3)x + (64728 - 544G3)x^2 + (51498 + 1272G3)x^3 + (16488 + 936G3)x^4 + (-5256 + 1008G3)x^5 + (-8424 - 768G3)x^6 + (-4446 + 186G3)x^7 + (-1152 + 174G3)x^8 + (-84 - 5G3)x^9 + (24 - 16G3)x^{10} + G3x^{11}$.*

Since cr_1, cr_2, and cr_3 are pre-images of $z = 1$, only the parameter planes of cr_i, where $i = 3, 4, \ldots, 13$, are obtained. The parameter plane represents whether, when taking an initial guess equal to the free critical point, its orbit converges or not. In this case, each point of the plane refers to a value of $G3 \in \hat{\mathbb{C}}$, that is, a member of Family (4). Instead of representing the 11 parameter planes, we just represent the unified parameter plane [25] for the sake of simplicity. White points represent convergence to any of the roots, whereas black points represent convergence to a different point or even divergence. The unified parameter plane is shown in Figure 4.



(**a**) (**b**)

Figure 4. Unified parameter plane: (**a**) complete, (**b**) a detail in $\mathrm{Re}\{G3\} \in [-100, 300]$, $\mathrm{Im}\{G3\} \in [-300, 300]$.

Let us remark on the broad white region in Figure 4. Taking a white point guarantees the selection of a stable method.

5. Numerical Performance

In this section, Model (6) and different academic problems are solved using the classical Rall's scheme and the iterative class (4), with the weight function described in (8) as $G'''(\mu) = 0$,

$$\begin{aligned} y_k &= x_k - \frac{2m}{2+m}\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \left(m + \frac{m^3(m-1)}{4}(\eta-\mu) + \right. \\ &\quad \left. \frac{m^4(2-2m-m^2+m^3)}{8(2+m)}(\eta-\mu)^2\right)\frac{f(x_k)}{f'(x_k)}, \quad k = 0,1,2,\ldots, \end{aligned} \tag{9}$$

where $\mu = \dfrac{m+2}{m}$. The calculations were performed in Matlab R2022b (9.13.0.2049777) using variable-precision arithmetic with 1000 digits of mantissa on a computer equipped with an Intel® Core™ i5-5200U CPU 2.20GHz. The tables that show the numerical performance collected the residuals $|x_{k+1} - x_k|$ and $|f(x_{k+1})|$ after convergence. Moreover, the computational estimation of the order of convergence $\tilde{\rho}$ [27] was obtained as

$$p \approx \tilde{\rho} = \frac{\ln(|f(x_{k+1})|/|f(x_k)|)}{\ln(|f(x_k)|/|f(x_{k-1})|)}, \quad k = 2,3,\ldots$$

When the components of vector $\tilde{\rho}$ did not tend to any real value, it is marked as '$-$'.

5.1. Model

To find the value of ε_{max} in (6), the equation was rewritten as

$$f(\varepsilon_{max}) = \varepsilon_{max} - \frac{f_y}{E_s} + \frac{f_{ct}A_c}{E_s A_s\left(1+\sqrt{3.6M\varepsilon_{max}}\right)}. \tag{10}$$

The values of the parameters were in the ranges displayed in Table 1.

Table 1. Ranges of the input parameters of Equation (6).

Input	Range
E_s(MPa)	[195,000, 205,000]
f_y(MPa)	[400, 500]
f_{ct}(MPa)	[25, 50]
ϕ(mm)	[6, 40]

Tables 2–4 show the numerical performance of the known Rall and Schröder's schemes and the proposed method (9) in solving Model (10). The stopping criterion set was $|f(x_{k+1})| < 10^{-16}$, taking as an initial guess, the value $x_0 = \frac{9}{4000}$.

As seen in Tables 2–4, for initial guesses close to the solution, the results for the proposed method show a clear improvement compared to the classical ones. The new method converged in two iterations to the solution for the used values of the parameters, with a clear improvement in the results provided by the classical methods. Let us remark that in Table 4, the value of $\tilde{\rho}$ cannot be displayed since the number of iterations was lower than three.

Table 2. Equation solved using Rall's method for $x_0 = \frac{9}{4000}$.

$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Solution**	**Iterations**	$\tilde{\rho}$
1.68025×10^{-18}	0.00000026893	0.000635603	7	1.8366
4.72168×10^{-19}	0.00000010345	0.000517160	8	2.1256
4.26835×10^{-20}	0.000000022056	0.000446156	8	1.9805
7.44847×10^{-17}	0.00000068368	0.000598853	7	1.9238
1.01531×10^{-17}	0.00000041925	0.000600227	7	1.9174
1.79118×10^{-19}	0.000000012939	0.000412873	8	1.9453
3.45775×10^{-19}	0.00000017295	0.000604944	7	1.9098
7.31454×10^{-17}	0.00000061574	0.000525394	8	2.2363
3.3872×10^{-19}	0.0000000014047	0.000537686	8	2.0056
2.15449×10^{-19}	0.00000000083436	0.000451730	8	2.0077
1.0075×10^{-17}	0.00000041298	0.000583335	7	1.9905
9.03263×10^{-20}	0.00000003903	0.000542714	8	2.0270
2.50559×10^{-19}	0.000000084226	0.000472623	8	2.0346
1.00944×10^{-18}	0.00000022828	0.000568953	7	2.0293
5.04157×10^{-18}	0.00000031256	0.000516531	8	2.1978

Table 3. Equation solved using Schröder's method for $x_0 = \frac{9}{4000}$.

$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Solution**	**Iterations**	$\tilde{\rho}$
4.13364×10^{-19}	0.00000014252	0.000635603	4	1.8842
2.02874×10^{-18}	0.00000023675	0.000517160	4	1.8483
3.83408×10^{-18}	0.00000028008	0.000446156	4	1.8264
9.91008×10^{-20}	0.00000016932	0.000598852	4	1.8738
7.63700×10^{-19}	0.00000016405	0.000600227	4	1.8751
5.66333×10^{-18}	0.000000030272	0.000412873	4	1.8142
2.77525×10^{-19}	0.0000001535	0.000604944	4	1.878
1.82188×10^{-18}	0.0000002397	0.000525394	4	1.8491
4.08786×10^{-19}	0.00000020586	0.000537686	4	1.8575
1.93359×10^{-18}	0.000000025206	0.000451730	4	1.8321
1.60503×10^{-19}	0.00000017179	0.000583335	4	1.8712
1.14606×10^{-18}	0.00000021945	0.000542714	4	1.8557
3.26226×10^{-18}	0.00000026803	0.000472623	4	1.8344
1.52974×10^{-19}	0.00000017252	0.000568953	4	1.869
1.53493×10^{-18}	0.00000024189	0.000516531	4	1.8473

Table 4. Numerical performance of Problem (10) for $x_0 = \frac{9}{4000}$.

$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Solution**	**Iterations**	$\tilde{\rho}$
2.95289×10^{-19}	0.0000138391	0.000635604	2	-
5.24395×10^{-19}	0.0000198302	0.000517161	2	-
1.46768×10^{-19}	0.0000207393	0.000446156	2	-
1.78237×10^{-19}	0.0000156739	0.000598853	2	-
5.2808×10^{-19}	0.0000151498	0.000600228	2	-
4.71609×10^{-20}	0.0000209598	0.000412874	2	-
1.09408×10^{-19}	0.000014158	0.000604944	2	-
2.25264×10^{-19}	0.0000205114	0.000525394	2	-
3.27036×10^{-19}	0.0000173992	0.000537687	2	-
1.94702×10^{-19}	0.0000180554	0.00045173	2	-
4.56095×10^{-19}	0.0000154389	0.000583336	2	-
1.24265×10^{-19}	0.0000190569	0.000542714	2	-
3.81289×10^{-19}	0.0000209862	0.000472623	2	-
1.46367×10^{-19}	0.0000150232	0.000568954	2	-
1.4193×10^{-19}	0.0000203555	0.000516532	2	-

Tables 5 and 6 show the effect of assuming an initial estimate further away from the root. In this case, $x_0 = \frac{1}{100}$, and the stopping criterion remained as in the previous case. The classical Rall's method did not converge using this initial estimation, but Schröder's scheme achieved convergence in a higher number of iterations compared to our proposed method.

Table 5. Equation solved using Schröder's method for $x_0 = \dfrac{1}{100}$.

$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Solution**	**Iterations**	$\tilde{\rho}$
2.94422×10^{-19}	0.0000000076303	0.000635603	5	1.959
4.70137×10^{-19}	0.0000000093136	0.000517160	5	1.9525
4.27242×10^{-20}	0.00000000010401	0.000446156	5	1.9466
1.81745×10^{-19}	0.0000000081385	0.000598852	5	1.9571
5.25711×10^{-19}	0.0000000080669	0.000600227	5	1.9572
1.79079×10^{-19}	0.000000011041	0.000412873	5	1.9431
1.08310×10^{-19}	0.0000000079021	0.000604944	5	1.9573
1.45071×10^{-19}	0.00000092512	0.000525394	5	1.9532
3.38714×10^{-19}	0.0000000089014	0.000537686	5	1.9535
2.15397×10^{-19}	0.000000010163	0.000451730	5	1.9464
4.59005×10^{-19}	0.0000000082459	0.000583335	5	1.9562
9.06579×10^{-20}	0.0000000089708	0.000542714	5	1.9541
2.69269×10^{-19}	0.00000000099794	0.000472623	5	1.9491
1.48503×10^{-19}	0.0000000083317	0.000568953	5	1.9553
2.15670×10^{-19}	0.0000000093495	0.000516531	5	1.9525

Table 6. Numerical performance of Problem (10) for $x_0 = \dfrac{1}{100}$.

$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Solution**	**Iterations**	$\tilde{\rho}$
2.9442×10^{-19}	0.00000168032	0.000635604	3	2.02379
4.70146×10^{-19}	0.00000684629	0.000517161	3	2.85371
4.27661×10^{-20}	0.00000803271	0.000446156	3	3.29596
1.81752×10^{-19}	0.00000260127	0.000598853	3	2.30058
5.25702×10^{-19}	0.00000220132	0.000600228	3	2.11224
1.78977×10^{-19}	0.00000883971	0.000412874	3	3.23475
1.0831×10^{-19}	0.00000160411	0.000604944	3	2.09545
1.45139×10^{-19}	0.00000858088	0.000525394	3	3.16438
3.3873×10^{-19}	0.00000347839	0.000537687	3	2.42474
2.15411×10^{-19}	0.00000336737	0.00045173	3	2.48239
4.59028×10^{-19}	0.00000223758	0.000583336	3	2.13696
9.06593×10^{-20}	0.00000589335	0.000542714	3	2.91293
2.69372×10^{-19}	0.00000885897	0.000472623	3	3.14682
1.48509×10^{-19}	0.00000182173	0.000568954	3	2.13834
2.15643×10^{-19}	0.00000799412	0.000516532	3	3.06402

The results in Table 6 show the good performance of the method, as it converged to the solution in three iterations. The value of $\tilde{\rho}$ differed from that of the theoretical one because in three iterations, it was not able to stabilize.

5.2. Academic Problems

From now on, we solve the following nonlinear equations with multiple roots:

- $\Phi_1(x) = \left(e^{-x} - 1 + \dfrac{x}{5}\right)^3$, whose root is $x^* \approx 4.9651$ of multiplicity $m = 3$, taking as an initial guess $x_0 = 26$,
- $\Phi_2(x) = x^2 e^x - \sin(x) + x$, whose root is $x^* = 0$ of multiplicity $m = 2$, taking as an initial guess $x_0 = 10$,
- $\Phi_3(x) = (x^2 - e^x - 3x + 2)^5$, whose root is $x^* \approx 0.2575$ of multiplicity $m = 5$, taking as an initial guess $x_0 = -6$.

The stopping criteria was set when $|f(x_{k+1})| < 10^{-200}$. The solution obtained using (9) was compared with the solutions applying Rall's (1) and Schröder's (2) methods, named O4, RA, and SC, respectively. Table 7 presents the results, where "nc" denotes that the method did not converge.

Table 7. Numerical performance of the problems $\Phi_1(x)$, $\Phi_2(x)$, and $\Phi_3(x)$.

Problem	Method	$\|f(x_{k+1})\|$	$\|x_{k+1} - x_k\|$	Iterations	$\tilde{\rho}$
$\Phi_1(x)$	RA	5.067×10^{-307}	1.5112×10^{-50}	6	2
	SC	4.4598×10^{-315}	6.8690×10^{-52}	6	2
	O4	0	7.2526×10^{-44}	4	3.9991
$\Phi_2(x)$	RA	2.0809×10^{-311}	2.7992×10^{-78}	15	2
	SC	nc	nc	nc	-
	O4	0	2.5147×10^{-65}	8	4.0000
$\Phi_3(x)$	RA	6.3688×10^{-289}	2.5494×10^{-29}	14	2
	SC	2.6371×10^{-307}	3.6995×10^{-31}	9	2
	O4	0	1.2030×10^{-40}	8	4.0004

The results in Table 7 illustrate the competitiveness of method O4 with respect to Rall's and Schröder's methods. In the three problems, the value of $|f(x_{k+1})|$ and the number of iterations improved compared to the classical methods. Moreover, the $\tilde{\rho}$ value matched that of the theoretical one.

6. Conclusions

In this paper, we developed a parametric family of fourth-order numerical methods for solving a constitutive equation of reinforced concrete (6) with multiple roots. A dynamical analysis was performed to select the best members of the family. For each method, its performance was compared with other known methods for multiple roots, obtaining the solution in fewer iterations with a higher order of convergence.

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3.3. Articulo 3

En este último artículo se presenta un método de tres pasos con orden de convergencia seis para resolver sistemas de ecuaciones no lineales. Este método numérico resulta especialmente eficiente, ya que tiene en su expresión iterativa un único operador inverso. Se compara con otros métodos existentes, resolviendo sistemas de ecuaciones académicos, y un sistema de ecuaciones proveniente de la ingeniería estructural, hormigón armado concretamente. Se tiene previsto realizar una presentación oral de este artículo, en la XXVI edición del congreso Modelling for Engineering & Human Behaviour 2024 (Valencia).

mathematics

Article

A Class of Efficient Sixth-Order Iterative Methods for Solving the Nonlinear Shear Model of a Reinforced Concrete Beam

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Abstract: In this paper, we present a three-step sixth-order class of iterative schemes to estimate the solutions of a nonlinear system of equations. This procedure is designed by means of a weight function technique. We apply this procedure for predicting the shear strength of a reinforced concrete beam. The values for the parameters of the nonlinear system describing this problem were randomly selected inside the prescribed ranges by technical standards for structural concrete. Moreover, some of these parameters were fixed taking into consideration the solvability region of the adopted steel constitutive model. The effectiveness of the new class is also compared with other current schemes in terms of the computational efficiency and numerical performance, with very good results. The advantages of this new class come from the low computational cost, due to the existence of an only inverse operator.

Keywords: nonlinear systems; iterative methods; reinforced concrete; shear behaviour; convergence order; efficiency

MSC: 65F10

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1. Introduction

Reinforced and prestressed concrete beams represent a structural type that resists internal stresses in a relatively complex manner due to their constitutive nature. Prior to the cracking of the concrete, the shear loads are carried by a set of diagonal compressive stresses complemented by another set of diagonal tensile stresses acting perpendicular to the first ones. Once the concrete tensile strength is reached, cracks form in the direction normal to the diagonal tensile stresses while pre-existing cracks spread and change inclination. Then, the ability of concrete to transmit diagonal tensile stresses is significantly reduced and the appropriate reinforcement is necessary to create a new system of internal stresses that carry the shear acting on the beam after cracking.

Between 1899 and 1902, Ritter [1] and Mörsch [2] proposed a truss model for explaining the field of forces in a cracked reinforced concrete beam, with the principal compressive stresses acting as diagonal members at 45° and the stirrups acting as vertical tension members. This model neglected the tensile stresses in the cracked concrete. In 1910, the first ACI Code modified Mörsch's 45° truss model through the addition of concrete in order to compensate for the conservatism of the model and to account for the fact that the crack angle is usually less than 45° [3]. Between 1904 and 1922, Talbot and Withey demonstrated that the stirrup stresses were lower than those predicted by the 45° truss model [4].

Thus, before using the equilibrium equations, the inclination of the diagonal compressive struts should be known. In 1929, Wagner [5] treated a similar problem by studying

the post-buckling shear response of thin metal girders, and he assumed that the angle of inclination of the diagonal tensile stresses coincided with the corresponding value of the diagonal tensile strains. Between 1974 and 1978, Collins and Mitchell developed a shear design model for reinforced concrete that, based on Wagner's assumption, predicted the inclination of the compressive struts by considering the strains in the transverse and longitudinal reinforcement, and in the diagonally stressed concrete [6,7]; this last approach became known as the Compression Field Theory (CFT).

In 1982, Vecchio and Collins found that the principal compressive stress in the concrete is a function not only of the principal compressive strain, but also of the coexisting principal tensile strain [8]. In 1986, they published, as a further development of the CFT, the so-called Modified Compression Field Theory (MCFT) [9], which accounts for the influence of the tensile stresses in the cracked concrete. This theory assumes a bilinear constitutive model for the steel and requires the checking of the local bar stresses at the cracks, ensuring thus that the smeared steel stresses between adjacent cracks are lower than the yield value.

Since the MCFT, other alternative approaches have been also developed due to the consideration given to other strength mechanisms, such as the dowel action of the reinforcement intersecting the cracks or the friction between the crack faces [10]. Likewise, different procedures have been proposed to treat the shear response of a reinforced concrete member in a continuum mechanics context (i.e., to account for tensile stresses in the diagonally cracked concrete), such as the developments of Hsu and his co-workers at the University of Houston [11,12], in the framework of the so-called Rotating-Angle Softened Truss Model (RA-STM). The RA-STM proposes a constitutive relationship for the reinforcement as stiffened by the concrete (i.e., the embedded model bar); due to this alternative concept, the steel stress does not exceed the yielding point and the local checking of cracks is no longer necessary. The last contribution in this line is the so-called Refined Compression Field Theory (RCFT) [13,14], where the embedded bar stress–strain relationship is obtained from the concrete tension stiffening model considered in the MCFT, so the crack check is avoided and a new formulation with respect to the traditional MCFT one is no longer needed. Moreover, the numerical results obtained from the RCFT lead to a better fitting of the experimental results, particularly in the region near the peak point in the shear–strain response, where MCFT significantly deviates from the experimental data. This last approach to the constitutive modeling of the steel reinforcement is the one considered in this work.

As it is justified in Section 2, CFT mechanical models involve several types of nonlinearities, among other reasons, due to the constitutive relationships of the reinforced concrete. These models usually require in their implementation the application of iterative fixed-point methods to solve the nonlinear systems that appear. In fact, the most efficient way to solve a nonlinear problem is usually to choose between accuracy and computational cost [15]. Moreover, in this work, the previous determination of a solvability region using algebraic procedures is also necessary in order to improve the efficiency of the numerical solver, as indicated in Section 2.

Solving systems of nonlinear equations is an important problem in science and engineering, as has been previously described. The objective is to find the roots of the nonlinear system $F(x) = 0$, F being a multidimensional function, $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, on D convex set, of size $n \times n$, $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T$ being f_i, $i = 1, 2, \ldots, n$, the functional coordinates of F.

One of the most commonly used methods is the classical Newton's method, which has a quadratic order of convergence and iterative expression:

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \quad k = 0, 1, 2, \ldots, \tag{1}$$

where $F'(x^{(k)})$ is the Jacobian matrix of F at k-th iteration.

Several Newton-type procedures, by using different techniques, have been published in the last few years, some introductory texts to this area are [16–20]. Their main aim is accelerating their convergence or increasing their efficiency with differently designed

techniques used by numerous authors in the literature (see, for example, [21–26]). In what follows, we are going to recall some of them for comparison purposes.

All the schemes we are going to mention use, in their iterative expression, the Jacobian matrix of function F and have, under the usual conditions, a convergence order 6. We will compare these methods, from the point of view of the results of the convergence order and computational efficiency, with the methods proposed in this paper that also have an order of 6 and only use $[F'(x)]^{-1}$ in their expressions.

In [27], by using the weight function procedure, the authors designed a Jarratt-type method for solving nonlinear systems, denoted by $M2_6$, the iterative expression of which is

$$\begin{cases} y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}),\ k = 0, 1, \ldots, \\ z^{(k)} &= x^{(k)} - \left(\frac{5}{8}I + \frac{3}{8}([F'(y^{(k)})]^{-1}F'(x^{(k)}))^2\right)[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= z^{(k)} - \left(\frac{-9}{4}I - \frac{15}{8}[F'(x^{(k)})]^{-1}F'(y^{(k)}) + \frac{11}{8}[F'(y^{(k)})]^{-1}F'(x^{(k)})\right)[F'(y^{(k)})]^{-1}F(z^{(k)}), \end{cases} \tag{2}$$

where I denotes the identity matrix of size $n \times n$. This method aims to evaluate the Jacobian matrix in two points and uses two inverse operators. These elements increase the number of operations per iteration.

In order to reduce the number of inverse operators, Narang et al. in [28], from a Chebyshev–Halley-type family, constructed a class of iterative schemes of the sixth order. One of its members, denoted by $M_{6,2}(1/2, 0)$, has the following iterative expression:

$$\begin{cases} y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} &= x^{(k)} - \left(\frac{1}{2}G(x^{(k)})\right)H(G(x^{(k)}))[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= z^{(k)} - \left(I + \frac{3}{2}G(x^{(k)})\right)[F'(x^{(k)})]^{-1}F(z^{(k)}),\ k = 0, 1, \ldots, \end{cases} \tag{3}$$

where $G(x^{(k)}) = I - [F'(x^{(k)})]^{-1}F'(y^{(k)})$ and $H(G(x^{(k)})) = I - \frac{1}{4}G(x^{(k)}) + \frac{11}{8}(G(x^{(k)}))^2$.

Behl et al. in [29], using the indeterminate parameter procedure, designed a family of iterative sixth-order methods for solving systems of nonlinear equations. One of its members, denoted by PM1, has the following iterative expression:

$$\begin{cases} y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} &= y^{(k)} - \left(4I - 3[F'(x^{(k)})]^{-1}F'(y^{(k)}) + \frac{9}{8}([F'(x^{(k)})]^{-1}F'(y^{(k)}))^{-2}\right)[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= z^{(k)} - \left(\frac{5}{2}I - \frac{3}{2}[F'(x^{(k)})]^{-1}F'(y^{(k)})\right)[F'(x^{(k)})]^{-1}F(z^{(k)}),\ k = 0, 1, \ldots \end{cases} \tag{4}$$

Finally, Yaseen and Zafar presented in [30] a Jarratt-type scheme of three steps for solving nonlinear systems, denoted by $FS6$, with sixth-order convergence and iterative expression:

$$\begin{cases} y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} &= x^{(k)} - \left(\frac{5}{8}[U_k]^{-1} + \frac{3}{8}U_k\right)[F'(y^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= z^{(k)} - \left(\frac{-13}{2}I + \frac{9}{2}[V_k]^{-1} + 3V_k\right)[F'(x^{(k)})]^{-1}F(z^{(k)}),\ k = 0, 1, \ldots, \end{cases} \tag{5}$$

where $U_k = [F'(y^{(k)})]^{-1}F'(x^{(k)})$ and $V_k = [F'(x^{(k)})]^{-1}F'(y^{(k)})$.

The rest of the paper is organized as follows. In Section 2, we describe the nonlinear system obtained for predicting the shear strength of a reinforced concrete beam. The efficient method for estimating its solution is presented in Section 3, as well as its convergence order. Section 4 is devoted to the efficiency analysis. The numerical performance of our proposed methods are studied on academical problems and on the nonlinear shear model described in Section 2. Finally, some conclusions are exposed.

2. Problem Statement

In [3], the authors proposed this stress–strain relationship for concrete cracked in tension:

$$\sigma_1 = \begin{cases} E_c \varepsilon_1, & \varepsilon_1 \leq \varepsilon_{ct}, \\ \dfrac{\alpha f_{ct}}{1 + \sqrt{500 \varepsilon_1}}, & \varepsilon_1 > \varepsilon_{ct}, \end{cases} \tag{6}$$

where σ_1 represents the contribution of tensile stresses in the concrete between the cracks or tension stiffening effect, ε_1 is the principal tensile strain, E_c being the modulus of elasticity of the concrete, ε_{ct} the strain related to the strength of the tensile, f_{ct}. Coefficient α is equal to 1.0 in cases of fast and non-cyclic loads and for deformed bars.

Regarding the concrete behaviour in compression, Vecchio and Collins formulated in [9], using the Modified Compression Field Theory (MCFT), the following relationship between diagonal compressive strain, ε_2, and the diagonal (or principal) compressive stress, σ_2:

$$\begin{aligned} &\sigma_2 = f_{2max} \left[2 \left(\tfrac{\varepsilon_2}{\varepsilon_c} \right) - \left(\tfrac{\varepsilon_2}{\varepsilon_c} \right)^2 \right], \\ &\text{with } \; f_{2max} = \tfrac{f_c}{0.8 + 170 \varepsilon_1} \leq f_c, \end{aligned} \tag{7}$$

where ε_c is the compressive stress related to the compressive strength of concrete in a cylindrical test f_c, f_{2max} is the maximum compressive stress in a diagonally cracked web and ε_1 is the coexisting principal tensile stress.

In CFT procedures, a perfect bond between concrete and steel is assumed; in consequence, any deformation developed by the reinforcement is identical to the one experienced by the surrounding concrete in the same direction; thus, a single average strain tensor of the composite material is adopted. The following relationship is considered regarding the compatibility of the strains in the reinforcement and the diagonally stressed concrete:

$$\tan^2 \theta = \frac{\varepsilon_x - \varepsilon_2}{\varepsilon_t - \varepsilon_2} = \frac{\varepsilon_1 - \varepsilon_t}{\varepsilon_1 - \varepsilon_x}, \tag{8}$$

where ε_x is the mean longitudinal strain and ε_t is the mean transversal strain on the web of a beam oriented according to the orthogonal $x - t$ direction (see Figure 1). The strain ε_2 is aligned in the direction of the compressive struts, at angle θ to the longitudinal axis (x) of the beam. Moreover, due to strain tensor, the main tensile strain is

$$\varepsilon_1 = \varepsilon_x + \varepsilon_t + \varepsilon_2. \tag{9}$$

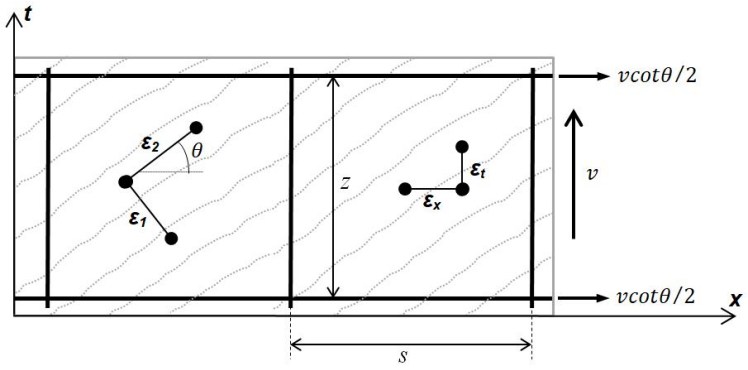


Figure 1. Strain compatibility between diagonally stressed concrete and the reinforcement in the cracked web of a reinforced concrete beam.

On the other hand, in CFT models, the equilibrium between the external loads and the internal forces is governed by the following equations:

$$\sigma_2 = \frac{\nu}{b_w z}(\tan\theta + \cot\theta) - \sigma_1, \tag{10}$$

$$2A_{st}\sigma_{st} = (\sigma_2 \sin^2\theta - \sigma_1 \cos^2\theta) b_w s, \tag{11}$$

$$4A_{sx}\sigma_{sx} + A_p\sigma_p = (\sigma_2 \cos^2\theta - \sigma_1 \sin^2\theta) b_w z = \frac{\nu}{\tan\theta} - \sigma_1 b_w z, \tag{12}$$

where θ is the angle of the main tensile stress, z is the flexural lever arm, s is the stirrup spacing, ν is the internal shear force, and b_w is the web width; A_{sx}, A_{st} and A_p are the cross-section surfaces for the longitudinal bars, the stirrup legs, and the prestressed reinforcement, respectively, and σ_{sx}, σ_{st} and σ_p are the related mean tensile stresses. The angles of the inclination of the principal strains coincide with the angles of the inclination of the principal stresses; this is known as EPA assumption or as Wagner's hypothesis [31].

Regarding the stress–strain relationship of the steel reinforcement, beyond the type of steel to consider (such as, for example, mild steel or stainless steel), CFT methods mainly differ in terms of the treatment of the steel behavior [9,11,13]. In this work, one of the most recent approaches to steel behaviour is adopted: the RCFT, previously introduced in Section 1, which is based on the concept of an embedded bar model that takes into account the concrete tension stiffening effect between cracks. The latter theory allows us to apply, in the most general case, the following mean stress–strain model for each type of steel reinforcement of the beam (i.e., longitudinal reinforcement and transverse stirrups):

$$\begin{aligned}
&\sigma_{s,i} = \begin{cases} f_{y,i} - \frac{\lambda_i A_{c,i}}{A_{s,i}} \frac{f_{ct}}{1+\sqrt{3.6 M_i \varepsilon_{s,i}}} & \text{if} \quad \varepsilon_{s,i} \geq \varepsilon_{max,i} \\ E_s \varepsilon_{s,i} & \text{if} \quad \varepsilon_{s,i} < \varepsilon_{max,i} \end{cases} \\
&i = \{x, t\} \\
&\text{in which} \\
&\varepsilon_{max,i} = \frac{f_{y,i}}{E_s} - \frac{\frac{\lambda_i A_{c,i} f_{ct}}{1+\sqrt{3.6 M_i \varepsilon_{max,i}}}}{E_s A_{s,i}} \\
&M_i = \frac{\lambda_i A_{c,i}}{\sum \pi \phi_i},
\end{aligned} \tag{13}$$

where the subscripts x and t refer to the longitudinal and the transverse reinforcement, respectively (then, (13) actually involves two equations); f_y is the steel yield stress, E_s is the elastic modulus of the steel, $\sigma_{s,av}$ is the average tensile stress in the steel, $\varepsilon_{s,av}$ is the average strain in the reinforcing bar, $\varepsilon_{max,i}$ is the apparent yield strain (cf. [13]), M is the joint parameter, A_s is the cross-section of the steel bars (longitudinal or transverse), A_c is the area of concrete attached to the bar that participates in the tensile stiffening effect; this is usually considered equal to the rectangular area surrounding the bar of diameter ϕ and over a distance no greater than 7.5ϕ from the center of the bar, and finally, λ_i is the coefficient for fixing the numerical solvability of the steel constitutive model.

In the case of prestressed concrete members, the following two additional equations are required:

$$\varepsilon_p = \varepsilon_x + \Delta\varepsilon_p, \tag{14}$$

$$\sigma_p = \begin{cases} E_p \varepsilon_p & , \quad \varepsilon_p \leq \frac{f_{py}}{E_p}, \\ f_{py} & , \quad \varepsilon_p > \frac{f_{py}}{E_p}, \end{cases} \tag{15}$$

where (14) represents the strain compatibility, $\Delta\varepsilon_p$ and ε_p being the strain imposed by the prestressing system and the strain of the prestressing strand, respectively, and Equation (15) represents the stress–strain relationship for the prestressing steel, f_{py} and E_p being its yield stress and elastic modulus, respectively.

Equation (13) is based on the concept of force equilibrium between a general section (or non-cracked section, where both the steel and the surrounding concrete contribute) and a cracked section (where only the reinforcement resists the internal forces; please see Figure 2). The greatest value of the area A_c in order to preserve the solvability of the embedded steel constitutive model proposed by the RCFT (i.e., in order to preserve the internal equilibrium of forces, in such a way that as the concrete participation increases, the steel stress diminishes) is obtained by the application of the following coefficient [32]:

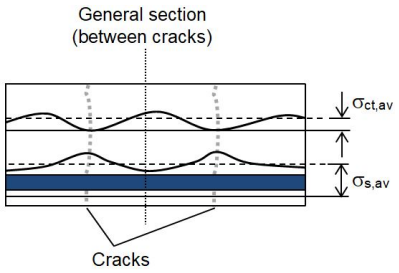


Figure 2. Average stress profiles ($\sigma_{ct,av}$ and $\sigma_{st,av}$) for an embedded reinforcement constitutive model including several cracks.

$$\lambda_{max,i} = \frac{A_s \cdot f_y}{A_c \cdot f_{ct}} \cdot \left(\frac{2}{3} + \frac{\sqrt{(1 + 10.8 \cdot M \cdot \epsilon_y)^3}}{48.6 \cdot M \cdot \epsilon_y} \right), \tag{16}$$

where the coefficient $\lambda_{max,i}$ represents the boundary of the solvability region for the embedded steel constitutive model in the i-direction (i.e., the maximum value of the coefficient λ in the i-direction in order to preserve the solvability), and ϵ_y is the strain corresponding to the steel yield stress (i.e., $\epsilon_y = f_y / E_s$). For certain design cases, the previous boundary may lay within the design range prescribed by technical codes for the tension stiffening area, A_c.

In summary, for a given value of tensile principal strain in concrete, ε_1, where such strain works as an input parameter, the shear model to predict the load–deformation response of a prestressed concrete beam is derived from the nonlinear system defined by (7)–(15), containing up to 10 equations (notice that (13) is actually two equations in turn) in the 10 unknowns (θ, ε_x, ε_t, ν, ε_2, σ_2, $\sigma_{s,x}$, $\sigma_{s,t}$, ε_p, and σ_p).

Two thousand solutions obtained from solving the nonlinear system of Equations (7)–(15) has been obtained from a set of input vectors uniformly generated. The range of the input parameters considered to this aim are presented in Table 1. These solutions were obtained using Newton's method and considering the same initial approximation of all the cases.

Table 1. Ranges for input parameters of the nonlinear system of Equations (7)–(15), with E_p = 190,000 MPa and $f_{py} = 1674$ MPa.

Input	Range
E_s(MPa)	[195,000, 205,000]
f_y(MPa)	[350, 500]
ϕ_x(mm)	[6, 40]
ϕ_t(mm)	[6, 40]
$\Delta\varepsilon_p(-)$	$[0.10 f_{py}/E_p, 0.90 f_{py}/E_p]$
$f_c(\text{mm}^2)$	[25, 50]
b_w(mm)	[100, 1000]
s(mm)	$[15\phi_t, 600]$
$A_p(\text{mm}^2)$	[300, 1200]
λ_x(mm)	$[0.1\lambda_{max,x}, 0.9\lambda_{max,x}]$
λ_t(mm)	$[0.1\lambda_{max,t}, 0.9\lambda_{max,t}]$
$\varepsilon_1(-)$	[0.0001, 0.01]

3. Development and Convergence of the Method

By using the weight matrix function procedure, we present a class of three-step iterative methods with the following iterative expression:

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} &= y^{(k)} - G(\mu^{(k)})b[F'(x^{(k)})]^{-1}F(y^{(k)})), \\ x^{(k+1)} &= z^{(k)} - G(\mu^{(k)})[F'(x^{(k)})]^{-1}\left(iF(z^{(k)}) + hF(y^{(k)})\right),\ k = 0, 1, 2, \ldots, \end{aligned} \tag{17}$$

where $\mu = [F'(x)]^{-1}F(y)$ is the variable of the weight function G, and b, i, and h are free parameters.

On the other hand, with F being a sufficiently differentiable Fréchet function, we can regard $\xi + m \in \mathbb{R}^n$ as being in the neighbourhood of the zero of F, ξ. Using Taylor developments and $F'(\xi)$ being nonsingular,

$$F(\xi + m) = F'(\xi)\left[h + \sum_{q=2}^{p-1} C_q m^q\right] + O(m^p), \tag{18}$$

where $C_q = \dfrac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$ for $q \geq 2$. Also, $C_q h^q \in \mathbb{R}^n$, as $F^{(q)}(\xi) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(\xi)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$. Therefore,

$$F(\xi + m) = F'(\xi)\left[I + \sum_{q=2}^{p-1} qC_q m^{q-1}\right] + O(m^{p-1}), \tag{19}$$

being $qC_q m^{q-1} \in \mathcal{L}(\mathbb{R}^n)$. For more details of this notation, see [33].

Indeed, following the notation introduced by Artidiello et al. in [34], the matrix function $G : X \to X$ can be defined in such a way that its Fréchet derivatives holds

(a) $G'(u)(v) = G_1 uv$, being $G' : X \to \mathcal{L}(X)$, $G_1 \in \mathbb{R}$
(b) $G''(u, v)(w) = G_2 uvw$, being $G_2 : X \times X \to \mathcal{L}(X)$, $G_2 \in \mathbb{R}$

when $X = \mathbb{R}^{n\times n}$ is the Banach space of real $n \times n$ matrices, and $\mathcal{L}(X)$ is the set of linear operators defined in X.

In the next result, we present the convergence of the family (17).

Theorem 1. *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differential Fréchet function defined on a convex neighborhood D of $\xi \in \mathbb{R}^n$, and a zero of F. Also, let $G : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be a sufficiently differentiable matrix function. Suppose that $F'(\xi)$ is nonsingular, and that $x^{(0)}$ is a seed sufficiently close to ξ. Therefore, the sequence $\{x^{(k)}\}_{k\geq 0}$ from (17) converges to ξ with an order of convergence six if $b = \dfrac{1}{G_0}$, $h = 0$, $i = \dfrac{1}{G_0}$, $G_1 = -G_0$, and $|G_2| < \infty$, $G_0 = G(I)$ and I being the identity matrix of size $n \times n$. In this case, the error equation is*

$$\begin{aligned} e^{(k+1)} = &\left(24C_2^5 - 4G_2(37C_2^5 - 6C_3 - 6C_2C_3C_2^2C_3C_2 + 3C_3C_2C_3)\frac{1}{G_0}\right. \\ &+4G_2(32C_2^5G_0 + C_2^5G_2 - 6G_0C_3C_2^3 - 6G_0C_2C_3C_2^2 - 6G_0C_2^2C_3C_2 + 3G_0C_3C_2C_3)\left(\frac{1}{G_0}\right)^2 \\ &\left.+2C_2^5G_0^2(3G_0 - G_2)\left(\frac{1}{G_0}\right)^3\right)e^{(k)^6} + O(e^{(k)^7}), \end{aligned}$$

where $e^{(k)} = x^{(k)} - \xi$ and $C_q = \dfrac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q = 2, 3, \ldots$

Proof. By means of the Taylor expansion of $F(x^{(k)})$ and $F'(x^{(k)})$ about ξ, we obtain

$$F(x^{(k)}) = F'(\xi)\Big[e^{(k)} + \sum_{i=2}^{6} C_i {e^{(k)}}^i\Big] + \mathcal{O}({e^{(k)}}^7),$$

and

$$F'(x^{(k)}) = F'(\xi)\Big[I + + \sum_{i=2}^{5} iC_i {e^{(k)}}^{i-1}\Big] + \mathcal{O}({e^{(k)}}^6).$$

We can deduce that

$$[F'(x^{(k)})]^{-1} = \Big[I + + \sum_{i=2}^{5} X_i {e^{(k)}}^{i-1}\Big][F'(\xi)]^{-1} + \mathcal{O}({e^{(k)}}^6),$$

where $X_2 = -2C_2$, $X_3 = -3C_3 + 4C_2^2$, $X_4 = -4C_4 + 6C_2C_3 + 6C_3C_2 - 8C_2^3$ and

$$\begin{aligned} X_5 &= -5C_5 + 8C_2C_4 - 12C_2^2C_3 + 9C_3^2 + 8C_4C_2 - 12C_2C_3C_2 + 16C_2^4 - 12C_3C_2^2, \\ X_6 &= -6C_6 + 10C_2C_5 + 12C_4C_3 - 18C_2C_3^2 - 18C_3C_2C_3 + 24C_2^3C_3 + 12C_3C_4 \\ &\quad - 16C_2^2C_4 + 10C_5C_2 - 16C_2C_4C_2 - 18C_3^2C_2 + 24C_2^2C_3C_2 - 16C_4C_2^2 \\ &\quad + 24C_2C_3C_2^2 + 24C_3C_2^3 - 32C_2^5. \end{aligned}$$

Then,

$$\begin{aligned} y^{(k)} - \xi &= C_2{e^{(k)}}^2 - 2(C_2^2 - C_3){e^{(k)}}^3 \\ &\quad - (4C_2C_3 + 3C_3C_2 - 4C_2^3 - 3C_4){e^{(k)}}^4 \\ &\quad - (-4C_5 + 6C_2C_4 - 8C_2^2C_3 + 6C_3^2 + 4C_4C_2 - 6C_2C_3C_2 + 8C_2^4 - 6C_3C_2^2){e^{(k)}}^5 \\ &\quad + \mathcal{O}({e^{(k)}}^6), \end{aligned}$$

and

$$(y^{(k)} - \xi)^2 = C_2^2{e^{(k)}}^4 + (2C_2C_3 + 2C_3C_2 - 4C_2^3){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6).$$

Moreover,

$$\begin{aligned} F(y^{(k)}) &= F'(\xi)\Big[C_2{e^{(k)}}^2 + 2(C_3 - C_2^2){e^{(k)}}^3 + (3C_4 + 5C_2^3 - 3C_3C_2 - 4C_2C_3){e^{(k)}}^4 + \\ &\quad (-12C_2^4 - 6C_3^2 + 4C_5 - 6C_2C_4 + 10C_2^2C_3 + 6C_3C_2^2 - 4C_4C_2 + 8C_2C_3C_2){e^{(k)}}^5\Big] \\ &\quad + \mathcal{O}({e^{(k)}}^6). \end{aligned}$$

So,

$$\begin{aligned} F'(y^{(k)}) &= F'(\xi)\Big[I + 2C_2^2{e^{(k)}}^2 + 4(C_2C_3 - 4C_2^3){e^{(k)}}^3 + (6C_2C_4 + 8C_2^2C_3 + 3C_3C_2^2){e^{(k)}}^4\Big] \\ &\quad + \mathcal{O}({e^{(k)}}^5), \end{aligned}$$

and the expansion of the variable $\mu^{(k)}$ is

$$\begin{aligned}\mu^{(k)} &= -2C_2e^{(k)} + (6C_2^2 - 3C_3){e^{(k)}}^2 + (-16C_2^3 - 4C_4 + 10C_2C_3 + 6C_3C_2){e^{(k)}}^3 \\ &+ (40C_2^4 + 9C_3^2 - 5C_5 + 14C_2C_4 - 28C_2^2C_3 - 15C_3C_2^2 + 8C_4C_2 - 18C_2C_3C_2){e^{(k)}}^4 \\ &+ (-96C_2^5 - 6C_6 - 30C_2C_3^2 + 18C_2C_5 - 40C_2^2C_4 + 72C_2^3C_3 + 36C_3C_2^3 + 12C_3C_4 \\ &- 12C_3^2C_2 - 24C_4C_2^2 + 12C_4C_3 + 10C_5C_2 + 42C_2C_3C_2^2 \\ &- 24C_2C_4C_2 + 48C_2^2C_3C_2 - 24C_3C_2C_3){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6).\end{aligned}$$

For the weight function G, we set

$$G(\mu^{(k)}) = G(I) + G_1(\mu^{(k)} - I) + \frac{1}{2}G_2(\mu^{(k)} - I)^2 + \mathcal{O}(\mu^{(k)} - I)^3,$$

that is,

$$\begin{aligned}G(\mu^{(k)}) &= G(I) - 2C_2G_1e^{(k)} + (6C_2^2G_1 - 3C_3G_1 + 2C_2^2G_2){e^{(k)}}^2 \\ &+ (-16C_2^3G_1 - 4C_4G_1 - 12C_2^3G_2 + 10G_1C_2C_3 + 3G_2C_2C_3 + 6G_1C_3C_2 \\ &+ 3G_2C_3C_2){e^{(k)}}^3 + \mathcal{O}({e^{(k)}}^4).\end{aligned}$$

We denote that $S = [F'(x^{(k)})]^{-1}F(y^{(k)})$. So, its Taylor development can be expressed as

$$\begin{aligned}S &= C_2{e^{(k)}}^2 + (-4C_2^2 + 2C_3){e^{(k)}}^3 + (13C_2^3 + 3C_4 - 8C_2C_3 - 6C_3C_2){e^{(k)}}^4 \\ &+ (-38C_2^4 - 12C_3^2 + 4C_5 - 12C_2C_4 + 26C_2^2C_3 + 18C_3C_2^2 - 8C_4C_2 + 20C_2C_3C_2){e^{(k)}}^5 \\ &+ \mathcal{O}({e^{(k)}}^6).\end{aligned}$$

So,

$$\begin{aligned}z^{(k)} &= (C_2 - bC_2G_0{e^{(k)}}^2 + (-2C_2^2 + 2C_3 + 4bC_3G_0){e^{(k)}}^3 \\ &+ (4C_2^3 + 3G_0 - 13bC_2^3G_0 - 3bC_4G_0 - 14bC_2^3G_2 - 2bC_2^3G_2 - 4C_2C_3 \\ &+ 8bGC_2C_3 + 4bG_1C_2C_3 - 3C_3C_2 + 6bG_0C_2 + 3bG_1C_3C_2){e^{(k)}}^4 \\ &+ (-8C_2^4 - 6C_3^2 + 38bC_2^4G_0 + 12bC_3^2G_0 + 66bC_2^4G_1 + 6bC_3^2G_1 + 20bC_2^4G_2 \\ &+ 4C_5 - 4bG_0C_5 - 6C_2C_4 + 12bG_0C_2C_4 + 6bG_1C_2C_4 + 8C_2^2C_3 - 26bG_0C_2^2C_3 \\ &- 28bG_1C_2^2C_3 - 4bG_2C_2^2C_3 + 6C_3C_2^2 - 18bG_0C_3C_2^2 - 18bG_1C_3C_2^2 - 3bG_2C_3C_2^2 \\ &- 4C_4C_2 + 8bG_0C_4C_2 + 4bG_1C_4C_2 + 6C_2C_3C_2 - 20bG_0C_2C_3C_2 - 22bG_1C_2C_3C_2 \\ &- 3bG_2C_2C_3C_2){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6),\end{aligned}$$

and

$$\begin{aligned}F(z^{(k)}) &= (C_2 - bC_2G_0){e^{(k)}}^2 + (-2C_2^2 + 2C_3 + 4bC_2G_0 - 2bC_3G_0 - 2bC_2^2G_1){e^{(k)}}^3 \\ &+ (5C_2^3 + 3C_4 - 15bC_2^3G_0 - 3bC_4G_0 + b^2c_2^3G_0^2 - 14bC_2^3G_1 - 2bC_2^3G_2 - 4C_2C_3 \\ &+ 8bG_0C_2C_3 + 4bG_1C_2C_3 - 3C_3C_2 + 6bG_0C_3C_2 + 3bG_1C_3C_2){e^{(k)}}^4 \\ &+ (-8C_2^4 - 6C_3^2 + 38bC_2^4G_0 + 12bC_3^2G_0 + 66bC_2^4G_1 + 6bC_3^2G_1 + 20bC_2^4G_2 + 4C_5 \\ &- 4bG_0C_5 - 6C_2C_4 + 12bG_0C_2C_4 + 6bG_1C_2C_4 + 8C_2^2C_3 - 26bG_0C_2^2C_3 \\ &- 28bG_1C_2^2C_3 - 4bG_2C_2^2C_3 + 6C_3C_2^2 - 18bG_0C_3C_2^2 - 18bG_1C_3C_2^2 - 3bG_2C_3C_2^2 \\ &- 4C_4C_2 + 8bG_0C_4C_2 + 4bG_1C_4C_2 + 6C_2C_3C_2 - 20bG_0C_3C_2 - 22bG_1C_2C_3C_2 \\ &- 3bG_2C_2C_3C_2){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6).\end{aligned}$$

Now, we denote $Sc = [F'(x^{(k)})]^{-1}F(z^{(k)})$. Therefore, its Taylor expansion is

$$\begin{aligned}
Sc = {} & (C_2 - bC_2G_0)e^{(k)^2} + (-4C_2^2 + 2C_3 + 6bC_2^2G_0 - 2bC_3G_0 + 2bC_2^2G_1)e^{(k)^3} \\
& + (13C_2^3 + 3C_4 - 27bC_2^3G_0 - 3bC_4G_0 + b^2C_2^3G_0^2 - 18bC_2^3G_1 - 2bC_2^3G_2 - 8C_2C_3 \\
& + 12bG_0C_2C_3 + 4bG_1C_2C_3 - 6C_3C_2 + 9bGC_3C_2 + 9bG_0C_3C_2 + 3bG_1C_3C_2)e^{(k)^4} \\
& + (-34C_2^4 - 12C_3^2 + 92bC_2^4G + 18bC_3^2G_0 - 2b^2C_2^4G_0^2 + 102bC_2^4G_1 + 6bC_3^2G_1 \\
& + 24bC_2^4G_2 + 4C_5 - 4bG_0C_5 - 12C_2C_4 + 18bG_0C_2C_4 + 6bG_1C_2C_4 + 24C_2^2C_3 \\
& - 50bG_0C_2^2C_3 - 36bG_1C_2^2C_3 - 4bG_2C_2^2C_3 + 18C_3C_2^2 - 36bG_0C_3C_2^2 - 24bG_1C_3C_2^2 \\
& - 3bG_2C_3C_2^2 - 8C_4C_2 + 12bG_0C_4C_2 + 4bG_1C_4C_2 + 18C_2C_3C_2 - 38bG_0C_2C_3 \\
& - 28bG_1C_2C_3C_2 - 3bG_2C_2C_3C_2)e^{(k)^5} + \mathcal{O}(e^{(k)^6}).
\end{aligned}$$

If $Ss = i \cdot Sc + h \cdot S$, then it is expanded as

$$\begin{aligned}
Ss = {} & (C_2h + C_2i - bC_2Gi)e^{(k)^2} \\
& + (-4C_2^2h + 2C_3h - 4C_2^2i + 2C_3i + 6bC_2^2G_0i - 2bC_3G_0i + 2bC_2^2G_1i)e^{(k)^3} \\
& + (13C_2^3h + 3C_4h + 13C_2^3i + 3C_4i - 27bC_2^3G_0i - 3bC_4G_0i + b^2C_2^3G_0^2i - 18bC_2^3G_0i \\
& - 2bC_2^3G_2i - 8hC_2C_3 - 8iC_2C_3 + 12bG_0iC_2C_3 + 4bG_1iC_2C_3 - 6hC_3C_2 - 6iC_3C_2 \\
& + 9bG_0iC_3 + 3bG_1iC_3C_2)e^{(k)^4} \\
& + (-38C_2^4h - 12C_3h - 34C_2^4i - 12C_3^2i + 92bC_2^4G_0i + 18bC_3^2G_0i - 2bC_2^4G_0^2i + 102bC_2^4G_0i \\
& + 6bC_3^2G_0i + 24bC_2^4G_2i + 4hC_5 + 4iC_5 - 4bGiiC_5 - 12hC_2C_4 - 12iC_2C_4 + 18bGiC_2C_4 \\
& + 6bG_1iC_2C_4 + 26hC_2C_3 + 24iC_2^2C_3 - 50bG_0iC_2^2C_3 - 36bG_1iC_2^2C_3 - 4bG_2iC_2^2C_3 \\
& + 18G_0C_3C_2^2 + 18iC_3C_2^2 - 36bGiC_3C_2^2 - 24bG_1iC_3C_2^2 - 3bG_2iC_3C_2^2 - 8hC_4C_2 - 8iC_4C_2 \\
& + 12bG_0iC_4C_2 + 4bG_1iC_4C_2 + 20hC_2C_3C_2 + 18iC_2C_3C_2 - 38bGiC_2C_3C_2 - 28bG_0iC_2C_3C_2 \\
& - 3bG_2iC_2C_3C_2)e^{(k)^5} + \mathcal{O}(e^{(k)^6}).
\end{aligned}$$

Then, the error equation is

$$\begin{aligned}
e^{(k+1)} = {} & (C_2 - bC_2G_0 - C_2G_0h - C_2G_0i + bC_2G_0^2i)e^{(k)^2} \\
& + (-2C_2^2 + 2d + 4bC_2^2G_0 - 2bdG_0 + 2bC_2^2G_0 + 4C_2^2G_0h - 2dG_0h \\
& + 2C_2^2G_0h + 4C_2^2G_0i - 2dG_0i - 6bC_2^2G_0^2i + 2bdG_0^2i + 2C_2^2G_1i - 4bC_2^2G_0G_1i)e^{(k)^3} \\
& + (4C_2^3 + 3C_4 - 13bC_2^3G_0 - 3bC_4G_0 - 14bC_2^3G_1 - 2bC_2^3G_2 - 13C_2^3G_0h - 3C_4G_0h \\
& - 14C_2^3G_1h - 2C_2^3G_2h - 13C_2^3Gi - 3C_4G_0i + 27bC_2^3G_0^2i + 3bC_4G_0^2i - b^2C_2^3G_0^3i \\
& - 14C_2^3G_1i + 36bC_2^3G_0G_1i + 4bC_2^3G_1^2i - 2C_2^3G_2ii + 4bC_2^3G_0G_2ii - 4C_2C_3 + 8bG_0C_2C_3 \\
& + 4bG_1C_2C_3 + 8G_0hC_2C_3 + 4G_1hC_2C_3 + 8G_0iC_2C_3 - 12bG_0^2iC_2C_3 + 4G_1iC_2C_3 \\
& - 8bG_0G_1iC_2C_3 - 3C_3C_2 + 6bG_0C_3C_2 + 3bG_1C_3 + 6G_0hC_3C_2 + 3G_1hC_3C_2 \\
& + 6G_0iC_3C_2 - 9bG_0^2iC_3C_2 + G_1iC_3C_2 - 6bG_0G_1iC_3C_2)e^{(k)^4} + M5e^{(k)^5} \\
& + M6e^{(k)^6} + \mathcal{O}(e^{(k)^7}).
\end{aligned}$$

By fixing $b = \frac{1}{G_0}$, $h = 0$, $i = \frac{1}{G_0}$, and $G_1 = -G_0$, the error equation becomes

$$\begin{aligned} e^{(k+1)} = \Bigg(& 24C_2^5 - 4G_2(37C_2^5 - 6C_3 - 6C_2C_3C_2^2C_3C_2 + 3C_3C_2C_3)\frac{1}{G_0} \\ & + 4G_2(32C_2^5G_0 + C_2^5G_2 - 6G_0C_3C_2^3 - 6G_0C_2C_3C_2^2 - 6G_0C_2^2C_3C_2 \\ & + 3G_0C_3C_2C_3)\left(\frac{1}{G_0}\right)^2 + 2C_2^5G_0^2(3G_0 - G_2)\left(\frac{1}{G_0}\right)^3\Bigg)e^{(k)^6} + \mathcal{O}(e^{(k)^7}). \end{aligned}$$

With this, the proof is finished. □

Let us notice that the order of convergence of this class of iterative methods can be increased up to 7 for specific values of G_2, depending on G_0 also being free. However, in order to reduce the computational cost, we set $G_2 = 0$, $G_0 = I$, and therefore the matrix weight function to be used in the iterative expression is

$$G(\mu^{(k)}) = 2I - \mu^{(k)} = 2I - [F'(x^{(k)})]^{-1}F'(y^{(k)}).$$

Therefore, the family is reduced to an iterative method of only order 6, denoted by O6, the iterative expression of which is

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}),\ k = 0, 1, 2, \ldots, \\ z^{(k)} &= y^{(k)} - \left[2I - [F'(x^{(k)})]^{-1}F'(y^{(k)})\right][F'(x^{(k)})]^{-1}F(y^{(k)}), \\ x^{(k+1)} &= z^{(k)} - \left[2I - [F'(x^{(k)})]^{-1}F'(y^{(k)})\right][F'(x^{(k)})]^{-1}F(z^{(k)}). \end{aligned}$$

Let us remark that this scheme has especially good properties, due to the existence of only one inverse operator. This yields that all the linear systems to be solved per iteration have the same coefficient matrix and therefore, the computational cost can be reduced by its LU factorization and the solution of several triangular linear systems. This is discussed in depth in the next section, in comparison with the introduced known procedures.

4. Efficiency Indices

To compare the iterative methods used, we use the computational efficiency index, CI, defined as [35]

$$CI = \rho^{\frac{1}{d + op}},$$

where d is the number of functional evaluations and op is the number of products/quotients per iteration.

In each iteration, five linear systems are solved with the same coefficient matrix, there are two matrix–vector products and, with respect to functional evaluations, we have two evaluations of Jacobian matrices and three of functions. The computational cost of method O6 is

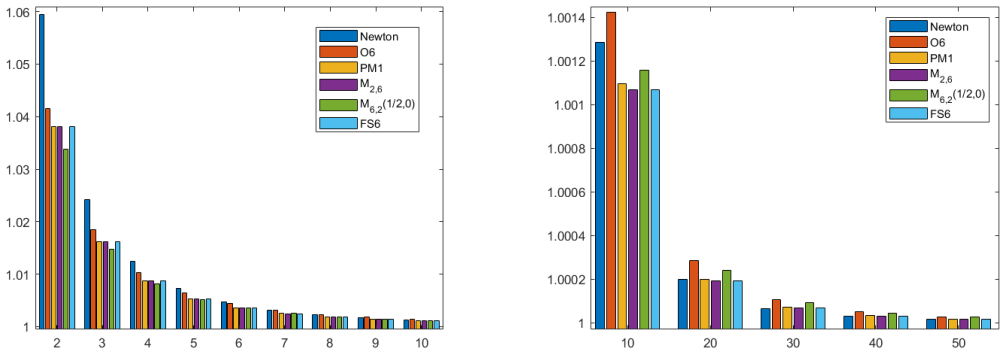
$$\frac{1}{3}n^3 + 9n^2 + \frac{8}{9}n, \tag{20}$$

In Table 2, the rest of the CI corresponding to the comparison methods are exposed. The way in which they have been calculated is similar to that of the O6 method.

Table 2. Comparisons of CI.

Method	CI
Newton	$2^{\frac{1}{\frac{1}{3}n^3+2n^2+\frac{2}{3}n}}$
O6	$6^{\frac{1}{\frac{1}{3}n^3+9n^2+\frac{8}{9}n}}$
PM1	$6^{\frac{1}{\frac{2}{3}n^3+11n^2+\frac{4}{3}n}}$
$M_{2,6}$	$6^{\frac{1}{\frac{2}{3}n^3+10n^2+\frac{4}{3}n}}$
$M_{6,2}(1/2,0)$	$6^{\frac{1}{\frac{1}{3}n^3+12n^2+\frac{5}{3}n}}$
FS6	$6^{\frac{1}{\frac{2}{3}n^3+10n^2+\frac{4}{3}n}}$

The results are represented in the semi-logarithmic scale; see Figure 3 for a better visualization of the differences between the indices (CI) for the methods used and several sizes (n) of the systems.



(**a**) Sizes from 2 to 10 (**b**) Sizes from 10 to 50 with a step of 10

Figure 3. Computational efficiency indices.

In Figure 3a, we can observe that, for $2 \leq n \leq 7$, the best CI index corresponds to the Newton method, O6 being the best for $n \geq 8$. In Figure 3b, we can check that for bigger systems, $n \geq 10$; the best CI remains as O6.

5. Numerical Performance

We analyze the performance of the methods described above to check their efficiency and compare it with other known methods. The results from Tables 3–6 correspond to the calculations made with Matlab R2022b, by using variable precision arithmetics with 1200 digits of mantissa, on a PC equipped with an *Intel* Core™i5-5200U CPU 2.20GHz. In all the tables, we show the residual errors $\|x^{(k+1)} - x^{(k)}\|$ and $\|F(x^{(k+1)})\|$ of the last iteration satisfying the stopping criterium $\|x^{(k+1)} - x^{(k)}\| < 10^{-300}$ or $\|F(x^{(k+1)})\| < 10^{-300}$, and the CPU time obtained as the mean of 20 executions (e-time). Moreover, a computational estimation of the order of convergence is obtained by the means of ACOC, introduced as

$$\rho \approx ACOC = \frac{\ln \frac{\|x^{(k+1)} - x^{(k)}\|}{\|x^{(k)} - x^{(k-1)}\|}}{\ln \frac{\|x^{(k)} - x^{(k-1)}\|}{\|x^{(k-1)} - x^{(k-2)}\|}}. \tag{21}$$

5.1. Example

We consider the nonlinear system, $F_1(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T = 0$, such that

$$f_i(x) = x_i - \cos\left(2x_i - \sum_{j=1}^{4} x_j - x_i\right), i = 1, 2, 3, 4 \ldots, 20, \tag{22}$$

with seed $x^{(0)} = (0.75, 0.75, \ldots, 0.75)^T$, and in this case, $\alpha \approx (0.519, 0.519, \ldots, 0.519)^T$.

Table 3. Numerical results for Example 5.1.

Method	Iteration	$\|x^{(k+1)} - x^{(k)}\|$	$\|F(x^{(k+1)})\|$	ρ	e-Time
Newton	8	3.1586×10^{-160}	2.297×10^{-320}	2.0	0.98
O6	4	3.4133×10^{-217}	0.0	6.0	0.99
FS6	4	1.682×10^{-201}	1.614×10^{-1207}	6.0	1.02
PM1	4	6.0584×10^{-186}	4.036×10^{-1115}	6.0	0.99
$M2_6$	4	4.7636×10^{-127}	1.891×10^{-635}	5.0	1.00
$M_{6,2}(1/2, 0)$	4	1.4631×10^{-189}	4.603×10^{-1137}	6.0	1.02

In Table 3, it can be observed that the number of iterations of all the sixth-order schemes are equal and the time is very similar in all of the methods; however, the best residual is obtained by the proposed scheme, O6. The ACOC estimates the theoretical order of the convergence accurately in all the cases.

5.2. Example

The second example is given by $F_2(x) = (g_1(x), g_2(x), \ldots, g_n(x))^T = 0$, such that

$$g_i(x) = x_i - 2\ln\left(1 + \sum_{j=1}^{n} x_j - x_i\right),\ i = 1, 2, \ldots, 20, \tag{23}$$

with seed $x^{(0)} = (1, 1, \ldots, 1)^T$ and $\alpha \approx (9.376, 9.376, \ldots, 9.376)^T$.

Table 4. Numerical results for Example 5.2.

Method	Iterations	$\|x^{(k+1)} - x^{(k)}\|$	$\|F(x^{(k+1)})\|$	ρ	e-Time (Sec)
Newton	11	1.1642×10^{-199}	3.409×10^{-401}	2.0	9.99
O6	5	3.3111×10^{-100}	1.032×10^{-608}	6.0	10.24
FS6	5	5.1171×10^{-73}	1.310×10^{-445}	6.0	11.00
PM1	6	2.169×10^{-291}	3.749×10^{-1755}	6.0	10.58
$M2_6$	6	5.3026×10^{-198}	4.327×10^{-996}	6.0	11.07
$M_{6,2}(1/2, 0)$	6	6.2633×10^{-289}	1.325×10^{-1206}	6.0	10.79

5.3. Example

Let us define now the nonlinear system $F_3(x) = (h_1(x), h_2(x), \ldots, h_n(x))^T = 0$, such that

$$h_i(x) = arctan(x_i) + 1 - 2\left(\sum_{j=1}^{n} x_j^2 - x_i^2\right),\ i = 1, 2, \ldots, n, \tag{24}$$

with seed $x^{(0)} = (0.5, 0.5, \ldots, 0.5)^T$, $n = 20$, and $\alpha \approx (0.1758, 0.1758, \ldots, 0.1758)^T$.

Of note, in Table 5, O6 and FS6 provide a solution satisfying the stopping criterium in a lower or equal number of iterations than the rest of the schemes. Indeed, the value of the residual errors in O6 and FS6 highly improve that of Newton's. This is the reason why their residuals are not as close to zero as those of the other schemes.

Table 5. Numerical results for Example 5.3.

Method	Iterations	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	ρ	e-Time (Sec)
Newton	10	1.2449×10^{-154}	1.322×10^{-307}	2.0	1.22
O6	5	1.3563×10^{-218}	2.414×10^{-1207}	6.0	1.24
FS6	4	4.4455×10^{-58}	4.283×10^{-344}	6.0	1.20
PM1	5	1.2252×10^{-218}	8.687×10^{-1208}	6.0	1.27
$M2_6$	5	6.2256×10^{-173}	5.016×10^{-861}	5.0	1.39
$M_{6,2}(1/2, 0)$	5	2.5983×10^{-252}	3.704×10^{-1208}	6.0	1.39

Regarding the applied problem described in Section 2, the underlying data of the nonlinear shear model of a reinforced concrete beam are provided by random values with few digits inside the prescribed ranges by technical standards for structural concrete; moreover, some of these parameters were fixed taking into consideration the solvability region of the adopted steel constitutive model. The stopping criterium is $\|x^{(k+1)} - x^{(k)}\| < 10^{-6}$ or $\|F(x^{(k+1)})\| < 10^{-6}$. The initial estimation used is $\theta = 34$, $\varepsilon_x = 0.0001$, $\varepsilon_t = 200,000$, $\nu = 0.0001$, $\varepsilon_2 = 0.0001$, $\sigma_2 = 0.0001$, $\sigma_{s,x} = 200$, $\sigma_{s,t} = 7$, $\varepsilon_p = 200$ and $\sigma_p = 100$. The results provided by the new and existing schemes appear in Table 6. The ACOC does not appear in this table, as it yields to unstable data in all cases.

Table 6. Problem statement Section 2.

Method	Iterations	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	e-Time (Sec)
Newton	5	0.0342	2.448×10^{-10}	18.3242
O6	3	9.0862	2.253×10^{-29}	21.6703
FS6	3	218.18	4.071×10^{-16}	22.1258
PM1	4	3.8554	3.138×10^{-28}	21.9594
$M2_6$	4	7.3926	9.621×10^{-21}	28.0727
$M_{6,2}(1/2, 0)$	3	0.0423	3.710×10^{-31}	22.4797

However, the best methods in terms of the number of iteration are O6, FS6, and $M_{6,2}(1/2, 0)$, all with three iterations. Among the sixth-order methods, the lowest e-time corresponds to our proposed scheme, O6. Although with this initial estimation, the e-time of Newton's method is the best, small changes in some of the coordinates of the seed yields to better results of O6 than Newton's scheme. This good performance allows us to assure the reliance and robustness of our proposed procedure.

6. Conclusions

In this article, we have developed a vectorial parametric family of numerical methods of the sixth order to solve nonlinear systems. In particular, it is applied on a constitutive equation of reinforced concrete (6). The order of the convergence of the new class (O6) is proven, and a particular member of the family is selected with better computational properties, as only one inverse operator is needed. Its efficiency is compared to other existing methods with the same order of convergence, and also with Newton's scheme, in terms of the computational efficiency index. For the size of the system $n \geq 8$, the proposed method, O6, gives the best results. In the numerical tests, all the comparison procedures need the same or more iterations and achieve lower precision results in the same or shorter execution time to achieve the required tolerance. This confirms the accuracy, robustness, and applicability of the proposed scheme.

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Capítulo 4

Resultados

4.1. Conclusiones

Aplicando las técnicas de composición y funciones peso, se han diseñado familias paramétricas de orden cuatro, para ecuaciones con raíces simples y múltiples, respectivamente, orden seis para estimar soluciones de sistemas de ecuaciones no lineales.

Se ha realizado el análisis complejo a los dos primeros métodos, dónde se han calculado sus puntos fijos y críticos. El análisis de las funciones asociadas ha permitido seleccionar los elementos más estables de sendas familias de métodos iterativos.

Se han implementado en MATLAB® todos los métodos, y se han representado sus planos dinámicos y de parámetros. Se han resuelto problemas abiertos de ingeniería de estructuras, escalares y vectoriales, utilizando las técnicas diseñadas, con resultados que mejoran los proporcionados por métodos existentes.

4.2. Futuras líneas de investigación

A partir del estudio de estos métodos y los resultados obtenidos, se proponen nuevas líneas de investigación, que detallamos a continuación.

- Diseñar nuevos métodos iterativos libres de derivadas, para raíces simples y múltiples, con mayores órdenes de convergencia, ópti-

mos, para resolver ecuaciones no lineales, más estables y eficientes.

- Extender la anterior línea de investigación, a sistemas de ecuaciones no lineales.
- Diseñar métodos iterativos con memoria, para estimar la solución de matrices inversas, raíces de matrices, y otras ecuaciones matriciales.
- Utilizar herramientas de dinámica real multidimensional para llevar a cabo un análisis cualitativo del esquema vectorial propuesto en esta memoria.

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Capítulo 5

Anexos

5.1. Códigos de programas

5.1.1. Programa Artículo 1

El siguiente código de MATLAB®, corresponde al programa ideas_a.m con el que se han realizado las pruebas numéricas del método iterativo para resolver ecuaciones no lineales, con orden de convergencia cuatro.

```
function [x,iter,incr1,incr2,ACOC] = ideas_a(f,x0,r,tol,maxiter)
format compact, digits 1000
%EJEMPLO: [x,iter,incr1,incr2,ACOC] = ideas_a('ecuacion',vpa(0.5),2,1e-
200,40)
incr=tol+1;
iter=0;
I=[];
while incr>tol && iter<maxiter
[fx,dfx]=feval(f,x0);
d=fx/dfx;
y=x0-(2/3)*d;
[~,dfy]=feval(f,y);
eta=dfy/dfx;
peso=(1)-(3/4)*(eta-1)+(1/2)*(9/4)*(eta-1)^2+r*(eta-1)^3;
x=x0-(peso*d);
iter=iter+1;
incr1=abs(x-x0);
I=[I incr1];
incr2=abs(feval(f,x));
incr=incr1+incr2;
x0=x;
end
ACOC=log(I(3:end)./I(2:end-1))./log(I(2:end-1)./I(1:end-2));
ACOC=vpa(ACOC,5);
incr1=vpa(incr1,5);
incr2=vpa(incr2,5);
end
```

5.1.2. Programa Artículo 2

El programa de MATLAB® que se muestra a continuación, O4_mult_2.m, corresponde al método iterativo para resolver ecuaciones no lineales, con raíces múltiples.

```
function [sol,iter,incre,incre2,ACOC] = O4_mult_2(fun,dfun,m,x0,tol,maxiter)
%EJEMPLO: [sol,iter,incre,incre2,ACOC] = O4_mult_2(@(x)(x^2-exp(x)-
3*x+2)^5,@(x)-5*(exp(x)-2*x + 3)*(3*x+exp(x)-x^2-2)^4,5,vpa(5),1e-10,40)
format compact; digits 200
incre2 = tol + 1;
iter = 0;
I=[];
[f]=feval(fun,x0);
while incre2>tol && iter<maxiter
[df]=feval(dfun,x0);
y =x0-(2*m)/(m+2)*f/df ;
[dfy] = feval(dfun,y);
h=(df/dfy)^(1/(m-1));
a=(2*m)/(2+m);
mu=-(m/(a-m));
G=m-(((a-m)^2*(-1+m)*m)/(a*(a-2*m+a*m)))*(h-mu)+1/2*(m^4*(2-2*m-
m^2+m^3))/(4*(2+m))*(h-mu)^2;
x=x0-G*f/df;
incre=abs(x-x0);
[f]=feval(fun,x);
incre2=abs(f);
I=[I,incre2];
iter=iter+1;
x0 = x;
end
ACOC=vpa(log(I(3:end)./I(2:end-1))./log(I(2:end-1)./I(1:end-2)),5);
sol=vpa(x,5);
incre=vpa(incre,5);
%end
end
function [f,df]=eq1(x)
f=x^3-(261*x^2)/50+(3633*x)/400-2107/400;
df=3*x^2-(261*x)/25+3633/400;
end
```

5.1.3. Programa Artículo 3

Este último programa, llamado orden6.m, corresponde al método iterativo para resolver sistemas de ecuaciones no lineales, con orden de convergencia seis.

```
function [t,p,ex1,ef1,iter] = orden6(f,x,tol,maxiter)
% [t,p,ex1,ef1,iter] = orden6('sistemas',vpa(0.75*ones(1,20)),1e-700,50)
digits 1200,format compact
ex1=tol+1;ef1=ex1;iter = 0;
X=[];EX=[];EF=[];P=[];
x=x(:);Inc=[];
while ex1>tol && ef1>tol && iter<maxiter
[fx,dfx]=feval(f,x);
y=x-dfx\fx;
[fy,dfy]=feval(f,y);
dy=dfx\fy;
z=y-2*dy+dfx\(dfy*dy);
[fz,~]=feval(f,z);
dz=dfx\fz;
t=z-2*dz+dfx\(dfy*dz);
iter=iter+1;
ft=feval(f,t);
ex1=norm(t-x);
ef1=norm(ft);
X=[X t];EX=[EX ex1];
x=t;
Inc=[Inc,ex1];
end
p=log(Inc(3:end)./Inc(2:end-1))./log(Inc(2:end-1)./Inc(1:end-2));
p=vpa(p,5);
ex1=vpa(ex1,5);ef1=vpa(ef1,5);


function [F,dF] = sistemas(x)
num=1;
x=x(:);
switch num
case 1
syms u v w
% sol=(0.90;0.66;1.57)
F1=cos(v)-sin(u);
F2=w^u-1/v;
F3=exp(u)-w^2;
F =[F1 F2 F3];%,pause
dy= jacobian(F,[u,v,w]);%,pause
u=x(1);
v=x(2);
w=x(3);
case 2
%solucion 0.57,0.57,0.57,-0.28
syms u v w t
F1=v*w+t*(v+w);
F2=u*w+t*(u+w);
F3=u*v+t*(u+v);
F4=u*v+u*w+v*w-1;
F =[F1 F2 F3 F4];%,pause
dy= jacobian(F,[u,v,w,t]);%,pause
u=x(1);
v=x(2);
w=x(3);
t=x(4);
```