

RESEARCH ARTICLE



# Pareto efficiency as alternative to Nash equilibrium in competitive discrete location under delivered pricing

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## ABSTRACT

We address the facility location problem for firms that sell an homogeneous product and compete on delivered prices. If the firms set equilibrium prices at each market, the problem can be seen as a location game for which there exists a Nash equilibrium if demand is inelastic. However, this equilibrium may be Pareto-inefficient and the firms could decide to collude if they can guarantee themselves higher payoffs than those prescribed by the equilibrium. The aim of this paper is to study the joint profit maximization problem to obtain a collusive solution of the location game (Pareto-efficient) and determine if such a solution is also a Nash equilibrium. We develop Integer Linear Programming models to find Pareto-efficient solutions for multiple and single facility location. Nash equilibria are obtained by the best response procedure. An empirical investigation is performed to compare the joint profit of the firms and the profit of each one of the firms, got through the Pareto-efficient locations, with those got through the Nash equilibrium.

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## 1. Introduction

Facility location choice is a strategic decision for firms that compete for demand with the aim of profit maximization. Apart from location, other variables have to be taken into account in order to maximize their profits, such as the price to be set in each market. A variety of location-price models have been studied in the literature. Many of them consider the location-price problem as a two stage game. In the first stage, the competing firms select their facility locations, in the second stage, they compete on price. The division into two stages is motivated by the fact that the choice of location is usually prior to the decision on price. The two-stage game is reduced to a single stage location game if there exists a unique price equilibrium in the second stage, which is determined by the set of facility locations chosen in the first stage. The existence of a unique price equilibrium often occurs when firms compete on delivered prices (Pacheco, 2009; Phlips, 1983), but it rarely exists if competence is on mill prices (Pelegrín & Pelegrín, 2017).

The location game under delivered pricing has extensively been studied for customers distributed on a linear segment, a circumference, or a circle, where firms will locate their facilities (D'Aspremont et al., 1979; De Palma et al., 1987; Gabszewicz & Thisse, 1992; Kilkenny & Thisse, 1999; Osborne & Pitchik, 1987). However, due to its complexity, it has been less studied for spatially markets where the

location space is the plane or a network. For completely inelastic demand and constant marginal production cost, Hoover (1937) analyzed the location game considering that each firm locates one facility, and determined that the equilibrium market price of a firm with the lowest delivery cost is equal to the next lowest delivery cost. This result was extended to a spatial duopoly on a compact subset of the plane by Lederer and Hurter (1986) and to a network by Lederer and Thisse (1990). In oligopoly, the same result was obtained by Dorta-González et al. (2005), who proposed a model where firms make location and delivery price decisions along a network of connected but spatially separated markets. Existence of a Nash Equilibrium (NE) for the location game has been proved in the above mentioned papers by showing that any global minimizer of the social cost is a NE. In Tardos and Wexler (2007), a similar location game is studied in the framework of potential games, taking social welfare as potential function. It is proved that any social welfare maximizer is also a NE. This result is equivalent to the one given in Dorta-González et al. (2005) taking into account that social cost minimization is equivalent to social welfare maximization in the location game studied in Tardos and Wexler (2007). On a network, there exists a global minimizer of the social cost at the nodes and the problem of minimizing the social cost has been solved when the firms locate multiple facilities by

using Mixed Integer Linear Programming (*MILP*) formulations (Pelegrín et al., 2011, 2017). On a plane, the problem of minimizing the social cost is harder to solve and it has been solved for two competing firms with single facility location (Díaz-Bañez et al., 2011; Fernández et al., 2014). If marginal production costs are not constant, or demand is price-sensitive, the socially optimal locations may not be a NE of the location game, as has been shown in Gupta (1994) and Hamilton et al. (1989), respectively. In such cases, to our knowledge, the existence of a pure strategy NE for the location game has not been proven yet.

The notion of NE is probably the most used solution concept in non-cooperative games (Eichberger, 1993; Tian, 2015). However, the NE as a first solution of such games has a negative property: there may exist a second solution such that the payoffs of each firm in the first solution are lower than the corresponding payoffs in the second one. This means that a NE may not be Pareto-efficient (PE), in which case the firms could cooperate to maximize their joint profit and allow to earn higher payoffs than their respective Nash payoffs. This cooperation usually requires bargaining among the firms to resolve their differences in order to select a mechanism to split the joint profit among the firms (collusive solution of the game). Although oligopoly collusion has extensively been studied in the economic literature, there is few research on how to solve the joint profit maximization problem under delivered pricing. Most of the papers analyze incentives to collude and stability of collusion taking a linear market as location space, where each firm locates a single facility (Andree et al., 2018; Gupta & Venkatu, 2002; Huck et al., 2003; Matsumura & Matsushima, 2005; Osborne & Rubinstein, 1990; Thisse & Vives, 1988). In this paper, we consider a discrete location space and face for the first time the joint profit maximization problem (*JPM*) to find PE solutions of the location game. The selection of a mechanism to split the joint profit among the competing firms is out of the scope of the paper.

Our main contributions are the Integer Linear Programming (*ILP*) models proposed to solve (*JPM*) for multiple and single facility location, respectively. First, however, we present an *ILP* model to find the optimal locations of one firm, assuming the location of its competitors are fixed. This model is implemented in the best response procedure to compute NE as alternative to social cost minimization. This procedure allows to determine if a PE solution is also a NE, or to generate a NE through a PE solution. An empirical study is performed to test the proposed models and to compare the profits

obtained by the competing firms when their facility locations are PE (cooperative solution) with the ones obtained when their facility locations are NE (noncooperative solution) generated through the PE locations.

The paper is organized as follows. In Section 2, the facility location game and notation are described. In Section 3, known results to determine NE are briefly presented. In Section 4, two formulations of (*JPM*) for multiple and single facility location with different and equal marginal delivered costs are presented. In Section 5, the proposed models are tested by solving a variety of test problems and an experimental investigation is performed to compare the profits obtained by each firm with Pareto and Nash locations. Finally, some conclusions are given in Section 6.

## 2. The facility location game

We consider a set of spatially distributed consumption markets and a set of firms which compete for demand of an homogeneous product with the aim of profit maximization. First, firms select the location of their facilities in a finite set of location candidates, then firms set delivered prices at each market. The customers in any market buy the product from the firm that offers the lowest price. It is assumed that the firms produce at a constant marginal delivered cost (it does not depend of the quantity produced), which make that markets can be treated independently for profit maximization if the locations of the firms are fixed.

The following notation will be used:

### Indices

|     |                                                   |
|-----|---------------------------------------------------|
| $i$ | index of firms; $i = 1, \dots, r$ .               |
| $j$ | index of location candidates; $j = 1, \dots, n$ . |
| $k$ | index of demand points; $k = 1, \dots, m$ .       |

### Data

|                                     |                                                                                                                |
|-------------------------------------|----------------------------------------------------------------------------------------------------------------|
| $F$                                 | set of firms.                                                                                                  |
| $L$                                 | set of location candidates.                                                                                    |
| $M$                                 | set of markets.                                                                                                |
| $q_k$                               | demand in market $k$ ; $k \in M$ .                                                                             |
| $f_i$                               | number of facilities of firm $i$ ; $i \in F$ .                                                                 |
| $pc_i(j)$                           | marginal production cost of firm $i$ at location $j$ ; $i \in F$ , $j \in L$ .                                 |
| $tc_i(j, k)$                        | marginal transportation cost of firm $i$ from location $j$ to market $k$ ; $i \in F$ , $j \in L$ , $k \in M$ . |
| $dc_i(j, k) = pc_i(j) + tc_i(j, k)$ | marginal delivered cost of firm $i$ from location $j$ to market $k$ ; $i \in F$ , $j \in L$ , $k \in M$ .      |

### Decision variables

|       |                                         |
|-------|-----------------------------------------|
| $X^i$ | set of facility locations of firm $i$ . |
|-------|-----------------------------------------|

## Miscellaneous

|                              |                                                                                                        |
|------------------------------|--------------------------------------------------------------------------------------------------------|
| $X = (X^1, X^2, \dots, X^r)$ | location strategy for the facilities of the competing firms.                                           |
| $X^{-i}$                     | vector of locations for the facilities of the firms other than $i$ ; $X = (X^i, X^{-i})$ , $i \in F$ . |
| $A$                          | set of facility locations, $A \subset L$ .                                                             |
| $C_k(A)$                     | minimum marginal delivered cost from the facilities in $A$ to market $k$ .                             |

It is assumed that the firms will not offer a price below the marginal delivered cost at each market. Ties in the lowest price offered at any market may occur and a tie breaking rule is needed. If two firms offer a lowest price at market  $k$ , the one with the minimum marginal delivered cost can lower its price and obtain all the demand at market  $k$ . If the tied firms have the same marginal delivered cost, no tie breaking rule is needed because they will obtain zero profit from that market as a result of price competition. Then we consider that ties in the lowest price are broken in favour of the firm with the minimum marginal delivered cost.

For any location strategy  $X$ , price competition leads to each firm monopolizing a group of markets from which the firm gets a positive profit. This group of markets not only depends on the locations of the facilities of firm  $i$ , but it also depends on the locations of the facilities of its competitors. Such group is denoted by  $M^i(X)$  and it is given by:

$$M^i(X) = \{k : C_k(X^i) < C_k(X^{-i})\}$$

Observe here that the sets  $M^i(X)$  for  $X$  fixed are pairwise disjoint, but do not necessarily cover  $M$ . Any market where the minimum marginal delivered cost at  $X$  is reached for at least two firms will belong to none of the  $M^i(X)$  (and no profit is to be gained there).

The price equilibrium at each market  $k$  depends on facility location. For completely inelastic demand, the equilibrium price for each firm  $i$  is:

$$\hat{p}_k^i(X) = \begin{cases} C_k(X^{-i}) & \text{if } C_k(X^i) < C_k(X^{-i}) \\ C_k(X^i) & \text{otherwise.} \end{cases}$$

If firms set the equilibrium prices, the profit obtained by each firm  $i$  is (Dorta-González et al., 2005):

$$\Pi^i(X) = \sum_{k \in M^i(X)} q_k (C_k(X^{-i}) - C_k(X^i))$$

Then the facility location problem for the competing firms can be seen as a non-cooperative location game. The firms are the players, the alternatives of player  $i$  are subsets  $X^i$  of the set of location candidates  $L$ , and the payoff that player  $i$  obtains is given by the profit function  $\Pi^i(X)$ ,  $i = 1, \dots, r$ .

## 3. Nash equilibrium

The most used solution concept for non-cooperative games is the NE. A location strategy  $\hat{X} = (\hat{X}^1, \hat{X}^2, \dots, \hat{X}^r)$  is a NE if for any  $i$  it is verified that:

$$\Pi^i(\hat{X}^i, \hat{X}^{-i}) \geq \Pi^i(X^i, \hat{X}^{-i}), \quad \forall X^i \subset L$$

The existence of a location NE for inelastic demand has been proved by using the concept of social cost (Dorta-González et al., 2005). The social cost is defined as the total cost incurred to supply demand to customers if each customer would pay for the product the minimum marginal delivered cost. For any fixed set of locations  $X = (X^1, X^2, \dots, X^r)$ , the social cost is:

$$S(X) = \sum_{k=1}^m q_k \min \{C_k(X^1), C_k(X^2), \dots, C_k(X^r)\}$$

It is verified that:

$$\Pi^i(X) = \sum_{k=1}^m q_k C_k(X^{-i}) - S(X)$$

From the previous expression follows that any global minimizer of  $S(X)$  is a location NE.

Social cost minimization is a problem hard to solve on a continuous location space. On a plane, it has only been solved when each firm locates a single facility (Fernández et al., 2014). In discrete location space and on a network with concave marginal production and transportation costs, the problem has been solved for multi-facility location by a Mixed Integer Linear Programming formulation (Pelegrín et al., 2011, 2017).

## 3.1. Best response procedure

An alternative to social cost minimization for finding a NE is the best response procedure. This procedure has extensively been used in Game Theory to find a NE if it exists (Eichberger, 1993). The procedure requires to solve a sequence of optimization problems. At each iteration there is a current location strategy  $X = (X^1, X^2, \dots, X^r)$ , and for each  $i$  the following optimization problem has to be solved:

$$BR^i(X^{-i}) : \text{Max} \{ \Pi^i(Y^i, X^{-i}) : |Y^i| = f_i, Y^i \subset L \}$$

which gives the best response of firm  $i$  to the locations of its competitors. The procedure is as follows:

*Best response procedure*

1. Select an initial set of facility locations  $X_0$ .

$$X_0 = (X_0^1, X_0^2, \dots, X_0^r), |X_0^i| = f_i, i = 1, \dots, r. \text{ Set } \nu = 0.$$

2. **For**  $i = 1, \dots, r$  **do**

Find an optimal solution  $X_{\nu+1}^i$  to problem

$BR^i(X_{\nu+1}^1, \dots, X_{\nu+1}^{i-1}, X_{\nu+1}^{i+1}, \dots, X_{\nu}^r).$

Set  $X_{\nu+1}^i = X_{\nu}^i$  if  $\prod_i (X_{\nu+1}^1, \dots, X_{\nu+1}^i, X_{\nu+1}^{i+1}, \dots, X_{\nu}^r) = \prod_i (X_{\nu+1}^1, \dots, X_{\nu+1}^{i-1}, X_{\nu}^i, \dots, X_{\nu}^r)$

**end for**

3. If  $X_{\nu+1}^i = X_{\nu}^i$ ,  $i = 1, \dots, r$ , a NE is found, **STOP**.  
Otherwise, set  $\nu = \nu + 1$  and **go to step 2**.

The previous procedure allow to determine if an initial set of facilities  $X_0$  is a NE. This occurs if and only if  $v(BR^i(X_0^{-i})) = \Pi^i(X_0^i, X_0^{-i})$  for all  $i = 1, \dots, r$ , where  $v(BR^i(X_0^{-i}))$  is the optimal value of problem  $BR^i(X_0^{-i})$ . Otherwise, the procedure generates a NE or it could cycle without finding a NE. The problem  $BR^i(X^{-i})$  is hard to solve in continuous location space, however if the set of location candidates is finite it can be formulated as an Integer Linear Programming problem as follows.

### 3.2. ILP formulation of $BR^i(X^{-i})$

For  $L = \{1, 2, \dots, n\}$  let us define the following sets and variables:

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$$\begin{aligned} L_k^i &= \{j \in L : dc_i(j, k) < C_k(X^{-i})\} \\ L^i &= \cup \{L_k^i : k \in M^i\} \\ M^i &= \{k : L_k^i \neq \emptyset\} \\ x_j^i &= \begin{cases} 1 & \text{if firm } i \text{ locates a facility at } j; j \in L^i \\ 0 & \text{otherwise} \end{cases} \\ w_{kj}^i &= \begin{cases} 1 & \text{if market } k \text{ is served by firm } i \text{ from location } j; k \in M^i, j \in L_k^i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$


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$L_k^i$  is the set of locations at which firm  $i$  can price below its competitors at market  $k$  and therefore get the full demand at  $k$ .  $L^i$  is the set of locations at which firm  $i$  can get a positive profit.  $M^i$  is the set of markets where firm  $i$  can get a positive profit. Variables  $x_j^i$  and  $w_{kj}^i$  are location and allocation variables, respectively.

If market  $k$  is served from location  $j \in L_k^i$ , the equilibrium price at  $k$  is  $\hat{p}_k^i(X) = C_k(X^{-i})$ . Then the profit maximization problem of firm  $i$  can be formulated as follows:

$$\begin{aligned} BR^i(X^{-i}) : \quad & \text{Max} \sum_{k \in M^i} \sum_{j \in L_k^i} q_k (C_k(X^{-i}) - dc_i(j, k)) w_{kj}^i \\ & \text{s.t.} \quad \sum_{j \in L^i} x_j^i = f_i \end{aligned} \quad (1)$$

$$w_{kj}^i \leq x_j^i, \quad j \in L_k^i, \quad k \in M^i \quad (2)$$

$$\sum_{j \in L_k^i} w_{kj}^i \leq 1, \quad k \in M^i \quad (3)$$

$$x_j^i, w_{kj}^i \in \{0, 1\}, \quad j \in L_k^i, \quad k \in M^i \quad (4)$$

The objective function of problem  $BR^i(X^{-i})$  represents the profit of firm  $i$ . Constraint (1) represents the number of facilities to be located by firm  $i$ .

Constraints (2) guarantee that variable  $w_{kj}^i$  may be positive only if firm  $i$  locates a facility at location  $j$ . Constraints (3) mean that each market  $k \in M^i$  can be served from at most one of the facilities of firm  $i$  (any of the facilities with the minimum marginal delivered cost in the optimal solution). Constraints (4) require that the variables are binary. The above problem is a Binary Integer Linear Programming (BILP) problem which contains a lot of binary variables. Observe that the optimal value of problem  $BR^i(X^{-i})$  does not change if variables  $w_{kj}^i$  are taken as non negative variables instead of binary variables. Replacing constraints  $w_{kj}^i \in \{0, 1\}$  by  $w_{kj}^i \geq 0$  in the above formulation, we obtain an equivalent problem which is a Mixed Integer Linear Programming (MILP) problem that could solve larger size problems. A formulation of  $BR^i(X^{-i})$  for price sensitive demand is given in García et al. (2011).

### 4. Pareto-efficiency

A location strategy  $\hat{X}$  is PE if there is no location strategy  $X$  such that  $\Pi^i(X) \geq \Pi^i(\hat{X})$  for any  $i \in F$  and  $\Pi^h(X) > \Pi^h(\hat{X})$  for at least one  $h \in F$ . This type of strategies corresponds to non-dominated strategies of the location game. Then acceptable NE as solutions of the location game are those which are PE. However, it is well known that a NE may not be PE, in such a case the firms could obtain a higher joint payoff than the one prescribed by the NE.

As alternative to the non-cooperative solution of the location game, we consider that the firms can cooperate and accept facility locations maximizing their joint profit (collusive solution). Note that a location strategy which is a maximizer of the joint profit is PE. Once a profit maximization solution  $X$  is obtained, we can use the best response procedure of subsection 3.1 to check if the strategy  $X$  is also a NE.

Facility locations which are both NE and PE are hard to find, sufficient and necessary conditions for a PE strategy to be a NE have been given for general non-cooperative games (Nessah & Parvulescu, 2017; Scalzo, 2010, 2012), but such conditions require convexity of the set of alternatives of each player and can not be applied to the proposed location game.

#### 4.1. Joint profit maximization

Each market  $k$  will be served by the facility with the minimum marginal delivered cost which offer a price at  $k$  equal to the minimum marginal delivered cost of its competitors. Given a location strategy  $X$  we denote by  $C_k^1(X)$  and  $C_k^2(X)$ , the minimum and the second minimum of the costs  $\{C_k(X^i) : i = 1, \dots, r\}$ , respectively, which might be equal. Then

the joint profit obtained for the location strategy  $X$  is given by:

$$\begin{aligned}\Pi(X) &= \sum_{i \in F} \Pi^i(X) = \sum_{i \in F} \sum_{k \in M^i(X)} q_k (C_k(X^{-i}) - C_k(X^i)) \\ &= \sum_{k=1}^m q_k (C_k^2(X) - C_k^1(X))\end{aligned}$$

Therefore, the collusive firms face the following optimization problem:

$$\begin{aligned}JPM : \quad \text{Max} \quad & \{\Pi(X) = \sum_{k=1}^m q_k (C_k^2(X) - C_k^1(X)) : \\ & |X^i| = f_i, X^i \subset L, i = 1, \dots, r\}\end{aligned}$$

In the following we develop two formulations of *JPM* in a discrete location space which will allow to find PE solutions to moderate size problems. The first one is for multi-facility location and firms with different or equal marginal delivered costs. The second is for the particular case of single facility location and firms with equal marginal delivered costs. We consider that at most one facility will be located at each location candidate so as to avoid zero profit facilities. Observe for instance that if transportation costs are equal for all facilities and two facilities are located at the same site, the one with the maximum production cost will obtain zero profit. On the other hand, locating such a zero profit facility at some other place will almost surely (if there are sufficiently many location candidates) yield non-zero profit at some other location.

#### 4.2. ILP formulation for the multi-facility JPM with different marginal delivered costs

To simplify notations we write  $C_{kj}^i = dc_i(j, k)$  for the marginal delivered cost of firm  $i$  from location  $j$  to market  $k$ . Any set of facility locations  $X = (X^1, X^2, \dots, X^r)$  can be represented by a multi-vector  $x = (x^1, x^2, \dots, x^r)$ , where  $x^i = (x_1^i, x_2^i, \dots, x_n^i)$  are binary vectors with components:

$$x_j^i = \begin{cases} 1 & \text{if location } j \in X^i \\ 0 & \text{otherwise.} \end{cases}$$

The minimum and second minimum marginal delivered costs at each market  $k$ ,  $C_k^1(X)$  and  $C_k^2(X)$ , will be determined through the following variables:

$$\begin{aligned}y_{kj}^i &= \begin{cases} 1 & \text{if } C_{kj}^i = C_k^1(X) \\ 0 & \text{otherwise.} \end{cases} \\ z_{kj}^i &= \begin{cases} 1 & \text{if } C_{kj}^i = C_k^2(X) \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

The firm with the minimum marginal delivered cost at market  $k$  will serve that market, and it can be determined by the following variables:

$$s_k^i = \begin{cases} 0 & \text{if firm } i \text{ serves market } k. \\ 1 & \text{otherwise.} \end{cases}$$

With those variables the joint profit maximization problem becomes the following Binary Non-linear Programming problem:

$$JPM_1 : \quad \text{Max} \quad \sum_{k=1}^m q_k \left[ \sum_{i=1}^r \sum_{j=1}^n C_{kj}^i (z_{kj}^i - y_{kj}^i) \right] \quad (1)$$

$$\text{s.t.} \quad \sum_{j=1}^n x_j^i = f_i \quad ; \quad \forall i \quad (2)$$

$$\sum_{i=1}^r x_j^i \leq 1 \quad ; \quad \forall j \quad (2)$$

$$y_{kj}^i + z_{kj}^i \leq x_j^i \quad ; \quad \forall i, \quad \forall j, \quad \forall k \quad (3)$$

$$\sum_{i=1}^r \sum_{j=1}^n y_{kj}^i = 1 \quad ; \quad \forall k \quad (4)$$

$$\sum_{i=1}^r \sum_{j=1}^n z_{kj}^i = 1 \quad ; \quad \forall k \quad (5)$$

$$\sum_{i=1}^r C_{kj}^i y_{kj}^i \leq \sum_{i=1}^r C_{kl}^i x_l^i + D_{kj} (1 - \sum_{i=1}^r x_l^i); \quad \forall j, \quad \forall l \neq j, \quad \forall k \quad (6)$$

$$\sum_{j=1}^n y_{kj}^i + s_k^i = 1 \quad ; \quad \forall i, \quad \forall k \quad (7)$$

$$\sum_{i=1}^r C_{kj}^i z_{kj}^i \leq \sum_{i=1}^r C_{kl}^i s_k^i x_l^i + D_{kj} (1 - \sum_{i=1}^r s_k^i x_l^i); \quad \forall j, \quad \forall l \neq j, \quad \forall k \quad (8)$$

$$x_j^i, s_k^i, y_{kj}^i, z_{kj}^i \in \{0, 1\}; \quad \forall i, \quad \forall j, \quad \forall k \quad (9)$$

The objective function gives the joint profit of the firms. Constraint (1) indicates the number of facilities to be located by each firm. Constraints (2) mean that at most one facility is located at each location candidate. Constraints (3) guarantee that the minimum and the second minimum marginal delivered cost to a market  $k$  can not be assigned to the same location and the same firm (take into account that ties in the minimum marginal delivered cost may occur). Constraints (4) means that for each market  $k$  only one firm and one location are selected with the minimum marginal delivered cost to a market  $k$ . Constraints (5) means that for each market  $k$  only one firm and one location are selected with the second minimum marginal delivered cost to a market  $k$ . Constraints (6) let determine the firm and the facility location at which the minimum marginal marginal delivered cost  $C_k^1(X)$  is obtained. Constraints (7) are used to identify the firm which serves market  $k$  (variable  $s_k^i = 0$ ) and those firms which do not serve market  $k$  (variable  $s_k^i = 1$ ) for each  $k \in K$ . Constraints (8) let determine the firm and the facility location at which the second minimum marginal delivered cost  $C_k^2(X)$  is obtained. Constraints (9) require that variables  $x_j^i$ ,  $s_k^i$ ,  $y_{kj}^i$  and  $z_{kj}^i$  are binary.  $D_{kj}$  in constraints (6) and

(8) is taken as  $D_{kj} = \max\{C_{kj}^i : i \in F\}$ , but it can be replaced by any fixed number greater than or equal to the maximum marginal delivered cost.

Constraints (8) in problem  $JPM_1$  are nonlinear. However they can easily be linearized by taking  $u_{kj}^i = s_k^i x_j^i$  and adding some additional constraints. Then the problem becomes the following Binary Linear Programming problem:

$$JPM_2 : \quad \text{Max} \quad \sum_{k=1}^m q_k \left[ \sum_{i=1}^r \sum_{j=1}^n C_{kj}^i (z_{kj}^i - y_{kj}^i) \right]$$

s.t. (1) to (7)

$$\sum_{i=1}^r C_{kj}^i z_{kj}^i \leq \sum_{i=1}^r C_{kl}^i u_{kl}^i + D_{kj} \left( 1 - \sum_{i=1}^r u_{kl}^i \right); \quad (8')$$

$$\forall j, \quad \forall l \neq j, \quad \forall k$$

$$u_{kj}^i \leq s_k^i; \quad \forall i, \quad \forall j, \quad \forall k \quad (10)$$

$$u_{kj}^i \leq x_j^i; \quad \forall i, \quad \forall j, \quad \forall k \quad (11)$$

$$s_k^i + x_j^i - u_{kj}^i \leq 1; \quad \forall i, \quad \forall j, \quad \forall k \quad (12)$$

$$x_j^i, s_k^i, y_{kj}^i, z_{kj}^i, u_{kj}^i \in \{0, 1\}; \quad \forall i, \quad \forall j, \quad \forall k \quad (9')$$

The previous formulations may become into mixed integer linear programming formulations. Because of (4) and (7) the  $s_k^i$  may be relaxed to continuous variables. They are automatically either 0 or 1. As explained in Plastria (2002) the  $u_{kj}^i$  may also be relaxed to continuous variables.

#### 4.3. ILP formulation for single-facility JPM with equal marginal delivered costs

If each firm locates one facility and marginal delivered costs are the same for all firms, the location game is symmetric. Then the joint profit maximization problem can be formulated in a simpler way as follows.

Any location strategy  $X = (X_1, X_2, \dots, X_r)$  is given by a set of  $r$  facility locations and can be represented by a binary vector  $x = (x_1, x_2, \dots, x_n)$  with components:

$$x_j = \begin{cases} 1 & \text{if location } j \in X \\ 0 & \text{otherwise.} \end{cases}$$

Let  $C_{kj}^i = C_{kj}$ , for  $i = 1, \dots, r$ , where  $C_{kj}$  denotes the marginal delivered cost from location  $j$  to market  $k$ . The minimum and second minimum marginal delivered costs  $C_k^1(X)$  and  $C_k^2(X)$  can be determined through the following variables:

$$y_{kj} = \begin{cases} 1 & \text{if } C_{kj} = C_k^1(X) \\ 0 & \text{otherwise.} \end{cases}$$

$$z_{kj} = \begin{cases} 1 & \text{if } C_{kj} = C_k^2(X) \\ 0 & \text{otherwise.} \end{cases}$$

With the previous variables the joint profit maximization problem becomes the following Binary Integer Linear Programming problem:

$$JPM_3 : \quad \text{Max} \quad \sum_{k=1}^m q_k \left[ \sum_{j=1}^n C_{kj} (z_{kj} - y_{kj}) \right] \quad (1)$$

$$\text{s.a.} \quad \sum_{j=1}^n x_j = r$$

$$y_{kj} + z_{kj} \leq x_j; \quad \forall j, \quad \forall k \quad (2)$$

$$\sum_{j=1}^n y_{kj} = 1; \quad \forall k \quad (3)$$

$$\sum_{j=1}^n z_{kj} = 1; \quad \forall k \quad (4)$$

$$C_{kj} y_{kj} \leq C_{kl} x_l + C_{kj} (1 - x_l); \quad \forall j, \quad \forall l \neq j, \quad \forall k \quad (5)$$

$$C_{kj} z_{kj} \leq C_{kl} (1 - y_{kl}) + C_{kj} (1 - x_l + y_{kl}); \quad \forall j, \quad \forall l \neq j, \quad \forall k \quad (6)$$

$$x_j, y_{kj}, z_{kj} \in \{0, 1\}; \quad \forall i, \quad \forall j, \quad \forall k \quad (7)$$

The objective function gives the joint profit of the firms. Constraint (1) indicates the number of facilities to be located, one per firm and all facilities have different locations. Constraints (2) guarantee that any location can not be both the location with the minimum and the second minimum marginal delivered cost to a market  $k$ . Constraints (3) means that for each market  $k$  only one facility is selected as the one with the minimum marginal delivered cost to a market  $k$ . Constraints (4) means that for each market  $k$  only one facility is selected as the one with the second minimum marginal delivered cost to a market  $k$ . Constraints (5) guarantee that  $y_{kj} = 1$  if  $j$  is the location with the minimum marginal delivered cost to a market  $k$ . Constraints (6) guarantee that  $z_{kj} = 1$  if  $j$  is the location with the second minimum marginal delivered cost to a market  $k$ . Constraints (7) require that variables  $x_j$ ,  $y_{kj}$  and  $z_{kj}$  are binary.

## 5. An empirical study

For testing the proposed models, we consider the municipalities in Spain of over 20,000 inhabitants as markets, and their demand being proportional to the population size of each municipality. The municipalities are numbered from 1 to 363 in decreasing population size, thus  $M = \{1, 2, \dots, 363\}$ . With the aim of knowing the differences in profit obtained for each solution when the set of location candidates increases, we have selected two sets of location candidates,  $L = \{1, 2, \dots, 10\}$  (municipalities of over 300,000 inhabitants) and  $L = \{1, 2, \dots, 15\}$  (municipalities of over 245,000 inhabitants). The sets  $M$  and  $L$  are represented in Figure 1. For simplicity, we consider zero production costs and marginal delivered costs being proportional to distances between facilities and markets. Then  $C_{kj}^i = \mu d(j, k)$ ,

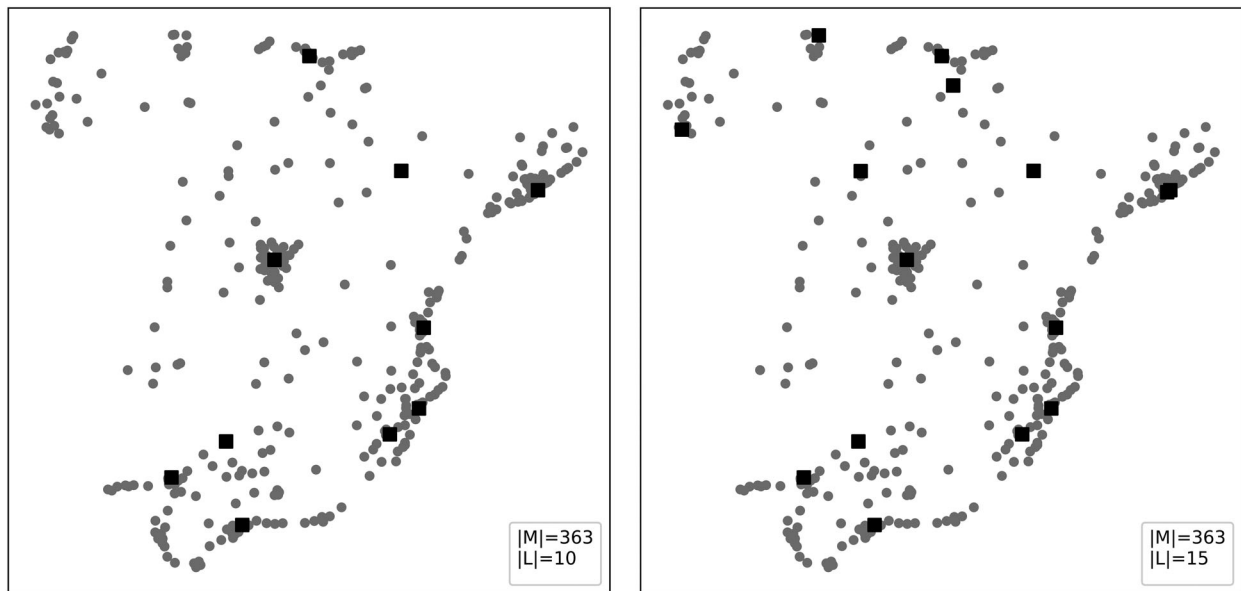


Figure 1. Demand points ○ and location candidates ■.

where  $\mu$  is a fixed value and  $d(j,k)$  have been approximated by using the Haversine formula to measure distances between two geographical points from their longitudes and latitudes (Mwemezi & Huang, 2011). The name, population size and geographical coordinates of the Spanish municipalities can be seen on the web: <http://www.um.es/geloca/gio/datos-espana-2015.txt>.

With the above choice of costs, the objective function of the joint profit maximization problem is proportional to  $\mu$ , so it is verified that the joint profit is increasing with  $\mu$  and the same optimal locations are obtained for any value of  $\mu$ . Therefore we have taken a fix value of  $\mu = 0.1$  for the computational experiments. On the other hand, we have symmetry: the firm numbers may be interchanged without modifying the joint profit.

Two computational experiments to compare the profits obtained by the competing firms with PE and NE solutions have been performed. The first, solving the model  $JPM_2$  to obtain PE solutions for two firms with multi-facility location in the range [2, 4]. The second, solving the model  $JPM_3$  to obtain PE solutions for single facility location and a number of firms  $r$  in the range [2, 6]. In both cases, NE have been determined by the best response procedure starting with the PE locations obtained from solving the corresponding  $JPM$  problem. The software FICO Xpress Mosel, 64bits v.3.10.0 for Linux, has been used to solve the proposed ILP problems on a computer with a processor Intel Core i7-6700 3.40 Ghzx8, RAM 8GB and OS Linux Ubuntu 15.10 64 bits.

The best response procedure depends on the order in which the firms are handled in step 2, so changing this order may lead to different NE,

probably with different joint profit. This is tested in the computational experiments. With the only two firms used in the first experiment (multi-facility) the permutations reduce to only two possibilities, so may easily be exhaustively tried. The presence of more firms in the second experiment (single facility) makes such complete enumeration prohibitive, so a random permutation selection approach is called for. To identify which firm number corresponds to each set of optimal facility locations, in the first experiment we consider firm 1 as the one with the smallest number of facilities. If the two firms have the same number of facilities, then firm 1 is assigned to the first set of optimal locations obtained when problem  $JPM$  is solved. In the second experiment, each one of the optimal locations is randomly assigned to each firm.

### 5.1. Multi-facility location

The results obtained from solving  $JPM_2$  are shown in Table 1. Columns 1, 2 and 3 correspond to the number of location candidates  $|L|$ , the number of facilities located by each firm  $(f_1, f_2)$  and the running time in minutes to find the profit maximizing locations  $X = (X^1, X^2)$ . Columns 4 and 5 show the optimal locations  $X^1$  and  $X^2$ , which are the PE solution to the corresponding location game. Columns 6 and 7 give the profits of the firms for the strategy location  $X = (X^1, X^2)$ , and column 8 shows the joint profit,  $\Pi(X) = \Pi_1(X) + \Pi_2(X)$ .

For any combination of  $[f_1, f_2]$ , as it was expected, the joint profit maximizing locations may change from taking  $|L| = 10$  to  $|L| = 15$ , and it is also verified that the joint profit does not decrease when  $L$  increases. However, it is surprising that the profit

of one of the firms may decrease when  $L$  increases. This happens for instance for  $[f_1, f_2] = [2, 4]$ , where the profit of firm 1 is 194 for  $|L| = 10$  and 95 for  $|L| = 15$ . Running times for  $|L| = 10$  are much smaller than the ones for  $|L| = 15$ , which is due to the number of constraints of  $JPM_2$  which notably increases with the size of  $L$ .

Starting with each one of the sets of locations  $X$  given in Table 1, we have applied the best response procedure changing the order the firms are handled and obtained a NE in all test problems. The results are shown in Table 2, where columns 3 and 4 show the order of the firms and the number of iterations in the best response procedure to obtain the NE. Column 5 gives the running time in seconds. Columns 6 and 7 show the locations  $Y^{(1)}$  and  $Y^{(2)}$  corresponding to the order the firms are handled, which are a NE of the respective location game. Columns 8 and 9 give the profits of the two firms for the strategy location  $Y = (Y^{(1)}, Y^{(2)})$ , and column 10 shows the joint profit,  $\Pi(Y) = \Pi_1(Y) + \Pi_2(Y)$ . Note that different NE are often obtained

when the order of firms is changed in the best response procedure (in 8 out of 12 test problems).

For any combination of  $[f_1, f_2]$ , contrary to what happens with the PE solution, it is not true for the NE solution that the joint profit increases when  $L$  increases. See for instance Table 2 for  $[f_1, f_2] = [2, 4]$ ,  $(X_2, X_1)$ , where the joint profit of the NE is 1254 for  $|L| = 10$  and 1022 for  $|L| = 15$ . Observe that now the running time to generate the NE for  $|L| = 10$  is similar to the one for  $|L| = 15$ , and it is less than 1 s in all cases.

The strategy location  $X$  is different from the strategy location  $Y$  in all test problems, which means that none of the PE solutions is also a NE of the corresponding location game. Note that the joint profit with the PE solution is greater than the one with the corresponding NE solution,  $\Pi(X) > \Pi(Y)$  (see Tables 1 and 2). This fact, could induce the firms to collude if difference in profit is high. The percentage of profit loss when moving from the PE locations to the corresponding NE locations is given by:

$$PL_i = 100 \left( 1 - \frac{\Pi_i(Y)}{\Pi_i(X)} \right), i = 1, 2$$

$$PL = 100 \left( 1 - \frac{\Pi(Y)}{\Pi(X)} \right)$$

**Table 1.** Pareto-efficient locations for two firms with multiple location.

| $ L $ | $[f_1, f_2]$ | $t(\text{min})$ | $X^1$    | $X^2$     | $\Pi_1(X)$ | $\Pi_2(X)$ | $\Pi(X)$ |
|-------|--------------|-----------------|----------|-----------|------------|------------|----------|
| 10    | [2,2]        | 2.26            | 2,8      | 4,6       | 741        | 419        | 1160     |
|       | [2,3]        | 3.47            | 4,6      | 1,2,3     | 194        | 992        | 1186     |
|       | [2,4]        | 2.90            | 4,6      | 1,2,3,8   | 194        | 1066       | 1260     |
|       | [3,3]        | 3.68            | 1,2,3    | 4,6,10    | 842        | 201        | 1043     |
|       | [3,4]        | 2.06            | 4,6,10   | 1,2,3,8   | 201        | 916        | 1118     |
|       | [4,4]        | 3.49            | 1,2,5,8  | 4,6,7,9   | 656        | 311        | 967      |
| 15    | [2,2]        | 22.30           | 2,14     | 4,12      | 546        | 645        | 1191     |
|       | [2,3]        | 26.79           | 2,14     | 1,4,7     | 241        | 1010       | 1251     |
|       | [2,4]        | 31.02           | 12,13    | 1,2,4,9   | 95         | 1234       | 1329     |
|       | [3,3]        | 57.34           | 1,2,3    | 4,6,12    | 887        | 251        | 1139     |
|       | [3,4]        | 51.11           | 4,6,12   | 1,2,3,15  | 242        | 951        | 1193     |
|       | [4,4]        | 93.88           | 1,2,3,15 | 4,6,10,12 | 835        | 249        | 1083     |

The results on profit loss are shown in Table 3, where columns 4 and 5 show the percentage of the profit loss for each firm, and column 6 gives the percentage of joint profit loss. A negative percentage means a percentage of profit gain.

The percentage of joint profit loss ( $PL$ ) is always positive, but the percentage of profit loss for one of the firms, ( $PL_1$  or  $PL_2$ ), may be negative. In such a case, the Pareto solution to the location game does not dominates the NE solution and one of the firms

**Table 2.** Nash equilibrium locations for two firms with multiple location.

| $ L $ | $[f_1, f_2]$ | $BP^i \text{ order}$ | $It$ | $t(\text{sec})$ | $Y^{(1)}$   | $Y^{(2)}$   | $\Pi_1(Y)$ | $\Pi_2(Y)$ | $\Pi(Y)$ |
|-------|--------------|----------------------|------|-----------------|-------------|-------------|------------|------------|----------|
| 10    | [2,2]        | $(X_1, X_2)$         | 2    | 0.18            | 1, 2        | 4, 9        | 570        | 282        | 852      |
|       |              | $(X_2, X_1)$         | 3    | 0.20            | 1, 4        | 2, 9        | 554        | 375        | 929      |
|       | [2,3]        | $(X_1, X_2)$         | 3    | 0.08            | 4, 8        | 1, 2, 9     | 228        | 630        | 857      |
|       |              | $(X_2, X_1)$         | 2    | 0.16            | 1, 2, 9     | 4, 8        | 630        | 228        | 857      |
|       | [2,4]        | $(X_1, X_2)$         | 2    | 0.05            | 4, 7        | 1, 2, 3, 8  | 195        | 717        | 912      |
|       |              | $(X_2, X_1)$         | 2    | 0.26            | 1, 2, 8, 9  | 4, 6        | 1069       | 184        | 1254     |
|       | [3,3]        | $(X_1, X_2)$         | 2    | 0.05            | 1, 2, 9     | 4, 6, 8     | 594        | 259        | 853      |
|       |              | $(X_2, X_1)$         | 2    | 0.04            | 4, 7, 8     | 1, 2, 3     | 270        | 472        | 742      |
|       | [3,4]        | $(X_1, X_2)$         | 2    | 0.04            | 4, 6, 7     | 1, 2, 3, 8  | 224        | 717        | 941      |
|       |              | $(X_2, X_1)$         | 3    | 0.06            | 1, 2, 7, 8  | 3, 4, 6     | 537        | 200        | 737      |
|       | [4,4]        | $(X_1, X_2)$         | 2    | 0.03            | 1, 2, 3, 8  | 4, 5, 6, 7  | 412        | 242        | 654      |
|       |              | $(X_2, X_1)$         | 2    | 0.03            | 3, 4, 6, 7  | 1, 2, 5, 8  | 329        | 525        | 854      |
| 15    | [2,2]        | $(X_1, X_2)$         | 3    | 0.38            | 1, 2        | 4, 9        | 570        | 282        | 852      |
|       |              | $(X_2, X_1)$         | 3    | 0.49            | 1, 4        | 2, 9        | 554        | 375        | 929      |
|       | [2,3]        | $(X_1, X_2)$         | 2    | 0.30            | 2, 8        | 1, 4, 9     | 280        | 716        | 996      |
|       |              | $(X_2, X_1)$         | 2    | 0.31            | 1, 4, 9     | 2, 8        | 716        | 280        | 996      |
|       | [2,4]        | $(X_1, X_2)$         | 2    | 0.29            | 12, 15      | 1, 2, 4, 9  | 121        | 901        | 1022     |
|       |              | $(X_2, X_1)$         | 2    | 0.30            | 1, 2, 4, 9  | 12, 15      | 901        | 121        | 1022     |
|       | [3,3]        | $(X_1, X_2)$         | 2    | 0.18            | 1, 2, 9     | 4, 12, 15   | 571        | 274        | 845      |
|       |              | $(X_2, X_1)$         | 3    | 0.10            | 4, 12, 15   | 1, 2, 9     | 274        | 571        | 845      |
|       | [3,4]        | $(X_1, X_2)$         | 2    | 0.06            | 4, 7, 12    | 1, 2, 3, 15 | 243        | 628        | 871      |
|       |              | $(X_2, X_1)$         | 2    | 0.27            | 1, 2, 9, 15 | 4, 6, 12    | 954        | 232        | 1186     |
|       | [4,4]        | $(X_1, X_2)$         | 3    | 0.18            | 1, 2, 7, 15 | 3, 4, 6, 12 | 473        | 248        | 721      |
|       |              | $(X_2, X_1)$         | 2    | 0.05            | 4, 6, 7, 12 | 1, 2, 3, 15 | 271        | 628        | 899      |

**Table 3.** Profit loss for two firms with multiple location.

| $ L $ | $[X_1, X_2]$ | BR order     | $PL_1(\%)$ | $PL_2(\%)$ | $PL(\%)$ |
|-------|--------------|--------------|------------|------------|----------|
| 10    | [2,2]        | $(X_1, X_2)$ | 23,12      | 32,64      | 26,56    |
|       |              | $(X_2, X_1)$ | -32,16     | 49,37      | 19,93    |
|       | [2,3]        | $(X_1, X_2)$ | -17,04     | 36,50      | 27,73    |
|       |              | $(X_2, X_1)$ | 36,50      | -17,04     | 27,73    |
|       | [2,4]        | $(X_1, X_2)$ | -0,56      | 32,73      | 27,60    |
|       |              | $(X_2, X_1)$ | -0,33      | 5,260      | 0,53     |
|       | [3,3]        | $(X_1, X_2)$ | 29,47      | -28,60     | 18,27    |
|       |              | $(X_2, X_1)$ | -33,96     | 43,96      | 28,93    |
|       | [3,4]        | $(X_1, X_2)$ | -11,09     | 21,75      | 15,83    |
|       |              | $(X_2, X_1)$ | 41,45      | 0,42       | 34,06    |
|       | [4,4]        | $(X_1, X_2)$ | 37,21      | 22,28      | 32,41    |
|       |              | $(X_2, X_1)$ | -5,62      | 19,99      | 11,75    |
| 15    | [2,2]        | $(X_1, X_2)$ | -4,46      | 56,26      | 28,45    |
|       |              | $(X_2, X_1)$ | 14,19      | 31,21      | 21,98    |
|       | [2,3]        | $(X_1, X_2)$ | -16,27     | 29,13      | 20,38    |
|       |              | $(X_2, X_1)$ | 29,13      | -16,27     | 20,38    |
|       | [2,4]        | $(X_1, X_2)$ | -26,94     | 26,98      | 23,11    |
|       |              | $(X_2, X_1)$ | 26,98      | -26,94     | 23,11    |
|       | [3,3]        | $(X_1, X_2)$ | 35,66      | -9,00      | 25,80    |
|       |              | $(X_2, X_1)$ | -9,00      | 35,66      | 25,80    |
|       | [3,4]        | $(X_1, X_2)$ | -0,45      | 33,94      | 26,96    |
|       |              | $(X_2, X_1)$ | -0,37      | 4,22       | 0,56     |
|       | [4,4]        | $(X_1, X_2)$ | 43,36      | 0,34       | 33,48    |
|       |              | $(X_2, X_1)$ | -8,97      | 24,72      | 16,98    |

could disagree to collude. However, the NE may also be strictly dominated by the PE locations ( $PL_i > 0$ , for  $i = 1, 2$ ). This happens in 3 out of the 12 test problems for  $|L| = 10$  and 2 out of the 12 test problems for  $|L| = 15$ . In such cases, each one of the firms would prefer the joint profit maximizing solution (PE) to the non cooperative solution (NE).

### 5.2. Single facility location

For single facility location, the results are shown in Table 4. Columns 1 and 2 correspond to the number of location candidates  $|L|$  and the number of competing firms  $r$ . Column 3 shows the set  $X$  of PE locations obtained by solving  $JPM_3$ . For each value of  $r$ , 3 random permutations in the set  $X$  are generated to determine the order the firms are handled in the best response procedure. Column 4 shows  $Y$ , the NE that is obtained by applying the best response procedure starting with  $X$ . Columns 5 and 6 give the joint profit of the firms for the location strategies  $X$  and  $Y$ , respectively.

For any number of firms  $r$ , as it was expected, both the PE and the NE locations may change from taking  $|L| = 10$  to  $|L| = 15$ . However, while the joint profit for the PE solution increases from taking  $|L| = 10$  to  $|L| = 15$ , the joint profit for the NE solution may decrease (see Table 4 for  $r = 3$ , where the joint profit of the NE is 826 for  $|L| = 10$  and 824 for  $|L| = 15$ ). Contrary to what happened for multiple facility location, the same NE is generated for single facility location by changing the order the firm are handled for each  $r$ , but for  $r = 6$  and  $|L| =$

**Table 4.** Pareto-efficient and Nash equilibrium locations for single facility location.

| $ L $ | $r$ | $X$               | $Y$                | $\Pi(X)$ | $\Pi(Y)$ |
|-------|-----|-------------------|--------------------|----------|----------|
| 10    | 3   | 8, 2, 6           | 1, 2, 6            | 904      | 826      |
|       |     | 2, 6, 8           | 2, 1, 6            |          |          |
|       |     | 6, 2, 8           | 1, 2, 6            |          |          |
|       | 4   | 2, 6, 1, 9        | 2, 4, 1, 9         | 823      | 808      |
|       |     | 1, 6, 9, 2        | 1, 4, 9, 2         |          |          |
|       |     | 9, 1, 6, 2        | 9, 1, 4, 2         |          |          |
|       | 5   | 4, 7, 1, 2, 8     | 4, 9, 1, 2, 8      | 761      | 758      |
|       |     | 8, 2, 4, 7, 1     | 8, 2, 4, 9, 1      |          |          |
|       |     | 2, 1, 7, 8, 4     | 2, 1, 9, 8, 4      |          |          |
|       | 6   | 7, 6, 1, 8, 2, 4  | 9, 6, 1, 8, 2, 4   | 672      | 660      |
|       |     | 2, 1, 4, 7, 6, 8  | 2, 1, 3, 4, 7, 8   |          |          |
|       |     | 7, 8, 1, 4, 2, 6  | 9, 8, 1, 4, 2, 6   |          |          |
| 15    | 3   | 12, 14, 4         | 1, 14, 6           | 1028     | 824      |
|       |     | 14, 4, 12         | 1, 14, 6           |          |          |
|       |     | 4, 12, 14         | 1, 6, 14           |          |          |
|       | 4   | 2, 10, 12, 13     | 2, 1, 4, 9         | 882      | 808      |
|       |     | 2, 12, 13, 10     | 2, 1, 9, 4         |          |          |
|       |     | 10, 2, 13, 12     | 1, 2, 4, 9         |          |          |
|       | 5   | 2, 4, 7, 12, 1    | 2, 4, 9, 8, 1      | 809      | 758      |
|       |     | 7, 2, 12, 1, 4    | 9, 2, 8, 1, 4      |          |          |
|       |     | 12, 7, 2, 4, 1    | 8, 9, 2, 4, 1      |          |          |
|       | 6   | 1, 7, 4, 12, 8, 2 | 1, 9, 4, 12, 15, 2 | 794      | 765      |
|       |     | 7, 1, 12, 8, 2, 4 | 9, 1, 12, 15, 2, 4 |          |          |
|       |     | 8, 12, 1, 4, 7, 2 | 15, 12, 1, 4, 9, 2 |          |          |

**Table 5.** Profit loss for single-facility location.

| $ L $ | $r$ | $PL_1(\%)$ | $PL_2(\%)$ | $PL_3(\%)$ | $PL_4(\%)$ | $PL_5(\%)$ | $PL_6(\%)$ | $PL(\%)$ |
|-------|-----|------------|------------|------------|------------|------------|------------|----------|
| 10    | 3   | -57.32     | 3.68       | 55.57      |            |            |            | 8.63     |
|       |     | 3.68       | -3.80      | 32.66      |            |            |            | 8.63     |
|       |     | -3.80      | 3.68       | 32.66      |            |            |            | 8.63     |
|       | 4   | 0.00       | -0.45      | 6.66       | -7.55      |            |            | 1.76     |
|       |     | 6.66       | -0.45      | -7.55      | 0.00       |            |            | 1.76     |
|       |     | -7.55      | 6.66       | -0.45      | 0.00       |            |            | 1.76     |
|       | 5   | -11.70     | -5.80      | -0.29      | 11.42      | 0.00       |            | 0.39     |
|       |     | 0.00       | 11.42      | -11.70     | -5.80      | -0.29      |            | 0.39     |
|       |     | 11.42      | -0.29      | -5.80      | 0.00       | -11.70     |            | 0.39     |
|       | 6   | -9.65      | -11.45     | -0.27      | 0.00       | 11.42      | 0.00       | 1.87     |
|       |     | 35.95      | 6.67       | -10.81     | -35.31     | -25.01     | 0.71       | 7.51     |
|       |     | -9.65      | 0.00       | -0.27      | 0.00       | 11.42      | -11.45     | 1.87     |
| 15    | 3   | -235.66    | 52.83      | 53.44      |            |            |            | 19.91    |
|       |     | 27.04      | 30.08      | -44.50     |            |            |            | 19.91    |
|       |     | -8.15      | -44.50     | 52.83      |            |            |            | 19.91    |
|       | 4   | 47.22      | 16.43      | -592.53    | -58.79     |            |            | 8.39     |
|       |     | 47.22      | -1398.27   | -58.79     | 61.37      |            |            | 8.39     |
|       |     | 16.43      | 47.22      | -103.50    | -440.36    |            |            | 8.39     |
|       | 5   | 15.30      | -11.70     | -5.80      | -31.24     | 20.88      |            | 6.33     |
|       |     | -5.80      | 15.30      | -31.24     | 20.88      | -11.70     |            | 6.33     |
|       |     | -31.24     | -5.80      | 15.30      | -11.70     | 20.88      |            | 6.33     |
|       | 6   | 11.91      | -5.80      | -11.70     | -3.00      | -0.42      | 12.35      | 3.59     |
|       |     | -5.80      | 11.91      | -3.00      | -0.42      | 12.35      | -11.70     | 3.59     |
|       |     | -0.42      | -3.00      | 11.91      | -11.70     | -5.80      | 12.35      | 3.59     |

10. In such a case 2 different NE are generated by changing the order the firms are handled.

The percentage of profit loss for each firm ( $PL_i, i = 1, \dots, r$ ) and the percentage of joint profit loss ( $PL$ ) are shown in Table 5.  $PL$  is always positive, but the percentage of profit loss is always negative for at least one of the firms. This means that, for single facility location, the Pareto solution to the location game does not dominates the NE solution in each one of the test problems. On the other hand, the percentage of joint profit loss for single facility is smaller than the one for multi-facility location when the firms move from the PE solution to the NE solution.

## 6. Conclusions

The location game of firms that compete under delivered pricing has been described for spatially separated markets. Pareto efficiency has been studied as alternative to the usual Nash equilibrium approach. To our knowledge, the joint profit maximization problem to obtain a PE solution is solved for the first time in discrete location space for spatially separated markets. The proposed *ILP* models for multiple and single facility location allow to obtain a collusive solution of the game (PE) which normally is different from the non cooperative solution (NE). An *ILP* model has been implemented in the best response procedure which allow to generate a NE through a PE solution. We have compared the joint profit of the firms and the profit of each firm when they choose the PE solution and the NE solution that result from the PE solution if the firms would decide not to collude.

The empirical study shows that the cooperative solution (PE) is always different from the non cooperative solution (NE). The joint profit with the PE solution is often much higher than the joint profit obtained with the NE solution, which can induce the firms to collude. On the other hand, the profit of at least one of the firms with the NE solution may be higher than its profit with the PE solution. When this occurs, collusion would require some kind of compensation for that firm. In some of the test problems, on the contrary, the profit of each one of the firms with the PE solution is higher than the one with the NE solution and all the firms would prefer the PE solution to the NE solution.

It is observed that the percentage joint profit loss when moving from the PE locations to the corresponding NE locations for single facility location is smaller than the one for multi-facility location. For single facility location, the profit of at least one of the firms with the NE solution is higher than its profit with the PE solution in all test problems. However, for multi-facility location, the profit of each firm with the PE solution is higher than the one with the NE in some test problems. In view of these results, it seems that collusion is more likely to happen when firms locate multiple facilities. The selection of a mechanism to split the joint profit among the firms that guarantees to each firm a profit greater than the one with a NE solution is a matter of future research.

## Disclosure statement

No potential conflict of interest was reported by the authors.

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