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## **Exact equilibrium quantities in spatially separated markets under production constraints**

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### **Abstract**

We deal with the Cournot game with production constraints, where the competing firms deliver an homogeneous product to spatially separated markets, linear demand functions may be different at each market, and linear production costs are non-identical. Despite the simplicity of this setting and although the existence and uniqueness of a Nash equilibrium have been proved, the problem of finding the exact equilibrium quantities has not been solved in the literature yet. In this paper, we analytically determine the unique equilibrium quantities for the constrained game when some conditions are verified. The equilibrium quantities and profits are compared with those obtained for the traditional Cournot game.

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## 1. Introduction

The Cournot game has been extensively studied in different frameworks and models over the past years (see [1, 5, 7, 9, 18, 20, 23] and the references therein). This game is often a subgame of a variety of games found in location choice or retail competition (see for instance [2, 17, 20, 24, 26]). Recent applications of Cournot competition can be found in many markets, as it happens in supply chain management or power markets (see for instance [10, 12, 16, 22]). Most models consider a linear price sensitive demand and nonnegative quantities to be delivered by the competing firms to the markets. Under some mild conditions, it is well known the existence of a unique Nash equilibrium (see [13, 21]). If the cost functions are linear, then positive quantities are delivered from each firm to each market at equilibrium, which are known as the Cournot quantities (see [8, 25]).

When production constraints are introduced, it is obvious that the Cournot quantities are not a Nash equilibrium if the total Cournot quantity to be delivered for at least one of the firms is greater than the maximum quantity that the firm can produce. Existence and uniqueness of Nash equilibrium has been proven for the capacity constrained Cournot game when the set of feasible quantities is convex and the cost functions are convex (see [15]). As it happens in other noncooperative constrained games (see [11]), determination of the Nash equilibrium involves solving nonlinear optimization problems or variational equations, for which numerical algorithms have been used (see [4, 14, 19, 22]). The solutions to such problems are approximated solutions and convergence to a global solution is required. Then the resulting solution to the equilibrium problem is an approximated Nash equilibrium. The contribution of this paper is to find the exact Nash equilibrium when the cost functions are linear. We use the Karush-Kuhn-Tucker optimality conditions to find analytical expressions for the quantities and determine the Nash equilibrium if some conditions are verified. A comparison of the solutions for the constrained Cournot game and the traditional Cournot game is presented.

The rest of the paper is organized as follows. In Section 2, the traditional Cournot game is described. Basic hypotheses and notation, the equilibrium quantities, and the profits when the firms choose the equilibrium quantities, are shown. In Section 3, the Cournot game with production constraints is studied. Conditions under which the Nash equilibrium can be analytically determined are shown. The equilibrium quantities are found and the resulting profit for each firm is obtained. In

Section 4, a comparison of results for the two games is presented. Finally, some concluding remarks are given in Section 5.

## 2. The Cournot game

Let us consider a set of firms  $F = \{1, \dots, r\}$  with production facilities located at different points in a geographical region. The firms compete on the quantity of product to be delivered to a set  $M = \{1, \dots, m\}$  of spatially separated markets. Each firm has to select the quantity to be delivered to each market with the aim of profit maximization. This problem can be seen as a non cooperative game with  $r$  players and payoff given by the profit achieved by each firm once all the quantities are determined.

The following notation will be used:

### *Indices*

$i, h$  indices of firms;  $i, h = 1, \dots, r$ .

$k$  index of markets;  $k = 1, \dots, m$ .

### *Data*

$F = \{1, 2, \dots, r\}$  set of competing firms.

$M = \{1, 2, \dots, m\}$  set of markets.

$p_k(q) = \alpha_k - \beta_k q$  inverse demand function in market  $k$ ;  $k \in M$ .

$pc_i$  unit production cost of firm  $i$ ;  $i \in F$ .

$tc(i, k)$  unit transportation cost from firm  $i$  to market  $k$ ;  $i \in F$ ,  $k \in M$ .

$c_{ik} = pc_i + tc(i, k)$  unit delivered cost from firm  $i$  to market  $k$ ;  $i \in F$ ,  $k \in M$ .

### *Decision variables*

$q_{ik}$  quantity delivered by firm  $i$  to market  $k$ ;  $i \in F$ ,  $k \in M$ .

### *Miscellaneous*

$Q = (q_{ik})$  matrix of all quantities delivered by the firms to the markets.

$Q_k = q_{1k} + q_{2k} + \dots + q_{rk}$  total quantity delivered to market  $k$ ;  $k \in M$ .

$q_i = (q_{i1}, q_{i2}, \dots, q_{im})$  vector of quantities delivered by firm  $i$ ;  $i \in F$ .

The following assumptions are considered:

1. The firms sell an homogeneous product.
2. The profit of each firm in a market is independent of the profit obtained by the firm in other markets.
3. The unit delivered cost is independent of the quantity delivered.

Once the quantities  $Q$  are determined, the payoffs of the Cournot game can be obtained. The profit gained by firm  $i$  at market  $k$  is:

$$\Pi_{ik}(Q) = q_{ik}(p_k(Q_k) - c_{ik}),$$

and the total profit obtained by firm  $i$  in all markets is:

$$\Pi_i(Q) = \sum_{k=1}^m q_{ik}(p_k(Q_k) - c_{ik}).$$

### The equilibrium quantities

For each  $i$ , the total profit of firm  $i$  is a separable function in variables  $q_{ik}$ ,  $k = 1, \dots, m$ . Then each firm  $i$  maximizes its profit in market  $k$  by delivering the quantity  $q_{ik}$  for which  $\partial \Pi_{ik} / \partial q_{ik} = 0$ . Hence, for any market  $k$ , the following system of linear equations is obtained:

$$\alpha_k - \beta_k(q_{1k} + q_{2k} + \dots + q_{rk}) - \beta_k q_{ik} = c_{ik}, \quad i = 1, 2, \dots, r.$$

This system of equations has a unique solution with positive quantities if  $p_k(Q_k) > c_{ik}$  for all  $i$ . The solutions to the system of equations are the equilibrium quantities at market  $k$ , or Cournot quantities, which are given by:

$$q_{ik}^* = \frac{1}{(r+1)\beta_k}(\alpha_k + \sum_{h \neq i} c_{hk} - r c_{ik}), \quad i = 1, 2, \dots, r.$$

The condition  $p_k(Q_k) > c_{ik}$  for all  $i$  is equivalent to,

$$\alpha_k > (r+1)[\max\{c_{ik} : i = 1, 2, \dots, r\}] - \sum_{i=1}^r c_{ik}$$

We will assume that the previous inequality holds for all  $k$ . This is a reasonable assumption due to the relative smallness of each delivered unit to each market, as compared to the maximum price that customers in the market are willing to pay for the product (see [25]).

### The profits at equilibrium

If the firms deliver the equilibrium quantities  $q_{ik}^*$  to each market  $k$ , then the profit achieved by firm  $i$ ,  $i = 1, 2, \dots, r$ , is given by:

$$\Pi_i(Q^*) = \sum_{k=1}^m \Pi_{ik}(Q^*) = \sum_{k=1}^m \beta_k (q_{ik}^*)^2 = \frac{1}{(r+1)^2} \sum_{k=1}^m \frac{1}{\beta_k} (\alpha_k + \sum_{h \neq i} c_{hk} - r c_{ik})^2.$$

### 3. The Cournot game with production constraints

In this Section we will determine the Nash equilibrium for the Cournot game with production constraints under some conditions. We consider that firms have their production limited to given maximum quantities which are known. Let  $A_i$  be the maximum quantity that firm  $i$  can produce,  $i = 1, 2, \dots, r$ . Firm  $i$  can only deliver a vector of quantities  $q_i = (q_{i1}, \dots, q_{im})$  so that:

$$\sum_{k=1}^m q_{ik} \leq A_i$$

In this case the feasible set of quantities is convex and the cost functions are linear, then a unique Nash equilibrium exists ( see [15]). The profit maximization problem for firm  $i$  becomes the following constrained nonlinear optimization problem:

$$\text{Maximize } \Pi_i(Q) = \sum_{k=1}^m [\alpha_k - \beta_k (q_{1k} + q_{2k} + \dots + q_{rk}) - c_{ik}] q_{ik}$$

s.t.

$$\begin{aligned} \sum_{k=1}^m q_{ik} &\leq A_i \\ q_{ik} &\geq 0, k = 1, \dots, m \end{aligned}$$

where the vectors  $q_h = (q_{h1}, q_{h2}, \dots, q_{hm})$  are fixed, for  $h \neq i$ , and  $q_i = (q_{i1}, q_{i2}, \dots, q_{im})$  is the vector of variables.

A Nash equilibrium is given by an  $r \times m$  matrix  $Q$  such that each row  $q_i$  is a profit maximization vector of quantities for firm  $i$  when firm  $h$  produces quantities given by the row  $q_h$  for  $h \neq i$  (see [6] for more details on Nash equilibria). Observe that the objective function of the previous problem is a concave function in variables  $q_{ik}$ ,  $k = 1, \dots, m$ . Since constraints are linear, an optimal solution to that problem is determined by the Karush-Kuhn-Tucker (KKT) optimality conditions (see [3]). Such

optimality conditions are given by the following system of equations and inequations in variables  $u_i$  and  $q_{ik}$ ,  $k = 1, \dots, m$ :

$$-\alpha_k + \beta_k(q_{1k} + q_{2k} + \dots + q_{rk}) + c_{ik} + \beta_k q_{ik} + u_i = 0 \quad (1)$$

$$\sum_{k=1}^m q_{ik} \leq A_i \quad (2)$$

$$u_i(A_i - \sum_{k=1}^m q_{ik}) = 0 \quad (3)$$

$$u_i \geq 0, q_{ik} \geq 0 \quad (4)$$

Therefore, a Nash equilibrium can be obtained by finding a matrix  $\tilde{Q} = (\tilde{q}_{ik})$  and a vector  $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_r)$  which verify the K-K-T optimality conditions for  $i = 1, \dots, r$  and  $k = 1, \dots, m$ . In such a case, the rows of matrix  $\tilde{Q}$  would be equilibrium quantities.

### Determination of the quantities

For each market  $k$ , from equation (1) for  $i = 1, \dots, r$ , it follows that the total quantity delivered to market  $k$  is:

$$Q_k = \sum_{i=1}^r q_{ik} = \frac{1}{(1+r)\beta_k} (r\alpha_k - \sum_{i=1}^r c_{ik} - \sum_{i=1}^r u_i),$$

which implies that the quantity delivered from firm  $i$  to market  $k$  is given by:

$$q_{ik} = \frac{1}{(1+r)\beta_k} (\alpha_k + \sum_{h \neq i} c_{hk} - rc_{ik} + \sum_{h \neq i} u_h - ru_i).$$

Observe that, for each  $i$  and  $k$ , the previous equation can be rewritten as:

$$q_{ik} = q_{ik}^* + \frac{1}{(1+r)\beta_k} (\sum_{h \neq i} u_h - ru_i) \quad (1')$$

Then optimality condition (1) is equivalent to (1'). Therefore, at equilibrium the quantity delivered from  $i$  to  $k$  must be equal to the Cournot quantity plus a quantity that is depending on the vector of multipliers  $u = (u_1, u_2, \dots, u_r)$ .

### Production constraints

For each  $i$ , if the quantities in the production constraint (2) are replaced by the quantities given in (1'), it follows that:

$$\sum_{h \neq i} u_h - ru_i \leq \frac{1+r}{\sum_{k=1}^m \frac{1}{\beta_k}} (A_i - \sum_{k=1}^m q_{ik}^*)$$

This inequality can be simplified by defining,

$$\beta = \sum_{k=1}^m \frac{1}{\beta_k}$$

and

$$B_i = A_i - \sum_{k=1}^m q_{ik}^*, \quad i = 1, 2, \dots, r.$$

Therefore, optimality condition (2) for each  $i$  becomes:

$$\sum_{h \neq i} u_h - ru_i \leq \frac{1+r}{\beta} B_i \quad (2')$$

### Determination of a Nash equilibrium

From (1') it follows that the quantities are determined by the non negative multipliers  $u_i$ , which must verify conditions (2') and (3). Furthermore, those quantities must be nonnegative. We will distinguish between the following two possible cases:

**Case 1 :**  $B_i \geq 0, i = 1, 2, \dots, r.$

In this case, all firms can produce the Cournot quantities  $q_{ik}^*$ . If we take the multipliers  $\tilde{u}_i = 0, i = 1, 2, \dots, r$ , then conditions (2') and (3) hold, and the quantities in (1') are  $q_{ik} = q_{ik}^*$ . Thus, the pair  $\tilde{Q} = (q_{ik}^*)$  and  $\tilde{u} = 0$  satisfy the K-K-T optimality conditions. Therefore, the matrix of Cournot quantities is a Nash equilibrium for the game with production constraints.

**Case 2 :** There is at least one  $h$  with  $B_h < 0$ .

In this case, firm  $h$  cannot produce the quantities  $q_{hk}^*, k = 1, \dots, m$ . Then the matrix of Cournot quantities is not a Nash equilibrium for the game with production constraints.

Let  $r_0$  be the number of positive multipliers,  $r_0 \leq r$ , and  $I_0 = \{i : u_i > 0\}$ . From optimality condition (3) it follows that condition (2') must be an equality for  $i \in I_0$ , therefore:

$$\sum_{h \in I_0} u_h - (1+r)u_i = \frac{1+r}{\beta} B_i, \quad i \in I_0$$

**Proposition 1 :** It is verified that:

$$\sum_{i \in I_0} B_i < 0$$

*Proof :* From the previous equalities it follows that:

$$r_0 \sum_{h \in I_0} u_h - (1+r) \sum_{i \in I_0} u_i = \frac{1+r}{\beta} \sum_{i \in I_0} B_i.$$

Then it is obtained that:

$$\sum_{i \in I_0} u_i = \frac{1+r}{\beta(r_0 - r - 1)} \sum_{i \in I_0} B_i.$$

Since  $u_i > 0$  for all  $i \in I_0$  and  $r_0 \leq r$ , negativity of  $\sum_{i \in I_0} B_i$  is guaranteed.  $\square$

Observe that there is a unique set of multipliers verifying the system of equations corresponding to condition (2') for  $i \in I_0$ , which are given by:

$$\tilde{u}_i = \frac{1}{\beta} \left[ \frac{1}{(r_0 - r - 1)} \sum_{h \in I_0} B_h - B_i \right]$$

The question now is how to select the set  $I_0$  so that,  $\tilde{u}_i > 0$  if and only if  $i \in I_0$ .

**Proposition 2 :** If  $u_i = 0$  then  $B_i > 0$ .

*Proof :* If  $u_i = 0$ , from (1') it follows that:

$$\begin{aligned} q_{ik} &= q_{ik}^* + \frac{1}{(1+r)\beta_k} (\sum_{h \neq i} u_h - r u_i) = q_{ik}^* + \frac{1}{(1+r)\beta_k} \sum_{h \in I_0} \tilde{u}_h = \\ &= q_{ik}^* + \frac{1}{(1+r)\beta_k} \frac{1+r}{\beta(r_0 - r - 1)} \sum_{h \in I_0} B_h = q_{ik}^* + \frac{1}{\beta\beta_k(r_0 - r - 1)} \sum_{h \in I_0} B_h \end{aligned}$$

Then, condition (2) for index  $i$  becomes:

$$\sum_{k=1}^m q_{ik} = \sum_{k=1}^m q_{ik}^* + \frac{1}{(r_0 - r - 1)} \sum_{h \in I_0} B_h = A_i - B_i + \frac{1}{(r_0 - r - 1)} \sum_{h \in I_0} B_h \leq A_i$$

which is equivalent to:

$$\frac{1}{(r_0 - r - 1)} \sum_{h \in I_0} B_h \leq B_i$$

Since the left hand of the previous inequality is positive, it follows that  $B_i > 0$ .  $\square$

From Proposition 2, it follows that multiplier  $u_i$  must be positive if  $B_i \leq 0$ . If  $B_i > 0$ , then condition (3) is verified by taking  $u_i = 0$ . Taking into account the previous results, the following theorem is obtained.

**Theorem 1 :** *Under the following conditions:*

- (i)  $q_{ik}^* \geq -\frac{1}{\beta\beta_k} B_i$  if  $B_i < 0$ ,
- (ii)  $\frac{1}{r_0 - r - 1} \sum_{h \in I_0} B_h \leq B_i$  if  $B_i > 0$ ,

the matrix  $\tilde{Q} = (\tilde{q}_{ik})$  given by:

$$\tilde{q}_{ik} = \begin{cases} q_{ik}^* + \frac{1}{\beta\beta_k} B_i & \text{if } B_i \leq 0 \\ q_{ik}^* + \frac{1}{\beta\beta_k(r_0 - r - 1)} \sum_{h \in I_0} B_h & \text{if } B_i > 0, \end{cases}$$

where  $I_0 = \{h : B_h \leq 0\}$ , is a Nash equilibrium.

**Proof :** If  $I_0 = \{h : B_h \leq 0\}$ , then the multipliers :

$$\tilde{u}_i = \frac{1}{\beta} \left[ \frac{1}{(r_0 - r - 1)} \sum_{h \in I_0} B_h - B_i \right], i \in I_0$$

are positive multipliers since there is at least one  $h$  with  $B_h < 0$ . By taking  $\tilde{u}_i = 0$  if  $i \notin I_0$ , we obtain the vector of non negative multipliers  $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_r)$ , from which the quantities  $\tilde{q}_{ik}$  are determined by optimality condition (1').

Observe that optimality condition (2') holds for  $i \in I_0$ . If  $i \notin I_0$ , condition (2') becomes:

$$\sum_{h \neq i} \tilde{u}_h - r\tilde{u}_i = \sum_{h \in I_0} \tilde{u}_i = \frac{1+r}{\beta(r_0 - r - 1)} \sum_{h \in I_0} B_i \leq \frac{1+r}{\beta} B_i$$

which is equivalent to condition (ii). Then optimality condition (2') holds for  $i = 1, \dots, r$ .

For the vector of multipliers  $\tilde{u}$ , it is clear that optimality condition (3) holds, for  $i = 1, \dots, r$ .

Finally, note that  $\tilde{q}_{ik} > q_{ik}^* > 0$  if  $B_i > 0$ ,  $\tilde{q}_{ik} = q_{ik}^* > 0$  if  $B_i = 0$ , and  $\tilde{q}_{ik} \geq 0$  if  $B_i < 0$  and condition (i) holds. Then, under condition (i), all quantities  $\tilde{q}_{ik}$  are non negative and then optimality condition (4) holds for  $i = 1, \dots, r$  and  $k = 1, \dots, m$ .

In consequence, under conditions (i) and (ii), the pair  $\tilde{Q}$  and  $\tilde{u}$  verify the K-K-T optimality conditions. Therefore,  $\tilde{Q}$  is a Nash equilibrium.  $\square$

Since the Cournot quantities are positive, condition (i) holds for small values of  $-B_i$ . In particular, if  $\beta_k = 1$  for all  $k$ , the condition becomes:

$$\min \{q_{ik}^* : k = 1, \dots, m\} \geq -\frac{1}{m} B_i$$

which holds for a sufficiently large number of markets. Observe that condition (ii) is not required if  $B_i \leq 0$  for all  $i$ . Otherwise, condition (ii) holds when all positive  $B_i$  are large enough, which is equivalent to saying that  $A_i$  is large enough. Note that condition (ii) is equivalent to:

$$\frac{1}{r_0 - r - 1} \sum_{h \in I_0} B_h \leq \min \{B_i : B_i > 0\}$$

Note that, if none of the firms can produce a quantity greater than the corresponding total Cournot quantity ( $B_i \leq 0$  for all  $i$ ) and condition (i) holds, then the Nash equilibrium is given by the matrix  $\tilde{Q} = (\tilde{q}_{ik})$  with :

$$\tilde{q}_{ik} = q_{ik}^* + \frac{1}{\beta\beta_k} B_i$$

### The profits at equilibrium

The pair  $\tilde{Q}$  and  $\tilde{u}$  shown in Theorem 1, under conditions (i) and (ii), verifies the K-K-T optimality conditions. If the firms deliver the

equilibrium quantities  $\tilde{q}_{ik}$ , from optimality condition (1), it follows that the profit achieved by firm  $i$ ,  $i = 1, 2, \dots, r$ , is given by:

$$\Pi_i(\tilde{Q}) = \sum_{k=1}^m (\beta_k \tilde{q}_{ik} + \tilde{u}_i) \tilde{q}_{ik} = \sum_{k=1}^m \beta_k \tilde{q}_{ik}^2 + \tilde{u}_i \sum_{k=1}^m \tilde{q}_{ik}.$$

#### 4. Comparison of the two games

If conditions (i) and (ii) given in Theorem 1 are verified, then the equilibrium quantities can be analytically determined for the constrained Cournot game, as it happens for the traditional Cournot game. Observe that verification of such conditions depend on the values of the Cournot quantities and the values of the slack variables  $B_i$  corresponding to the production constraints when the quantities are the Cournot quantities. In such a case, at equilibrium it is obtained that:

- a) If  $B_i > 0$ , then firm  $i$  delivers a quantity greater than the Cournot quantity to each market,  $\tilde{u}_i = 0$ , and the profit obtained by the firm is given by:

$$\begin{aligned} \Pi_i(\tilde{Q}) &= \sum_{k=1}^m \beta_k \left[ q_{ik}^* + \frac{1}{\beta \beta_k (r_0 - r - 1)} \sum_{h \in I_0} B_h \right]^2 = \\ &= \sum_{k=1}^m \beta_k [q_{ik}^*]^2 + \frac{1}{\beta (r_0 - r - 1)} \left( \sum_{h \in I_0} B_h \right) \left( 2 \sum_{k=1}^m q_{ik}^* + \frac{1}{r_0 - r - 1} \sum_{h \in I_0} B_h \right) \end{aligned}$$

- b) If  $B_i = 0$ , then firm  $i$  delivers the Cournot quantity to each market,  $\tilde{u}_i > 0$ , and the profit obtained by the firm is given by:

$$\Pi_i(\tilde{Q}) = \sum_{k=1}^m \beta_k [q_{ik}^*]^2 + \frac{1}{\beta} \left[ \frac{1}{r_0 - r - 1} \sum_{h \in I_0} B_h \right] A_i$$

- c) If  $B_i < 0$ , then firm  $i$  delivers a quantity smaller than the Cournot quantity to each market,  $\tilde{u}_i > 0$ , and the profit obtained by the firm is given by:

$$\begin{aligned} \Pi_i(\tilde{Q}) &= \sum_{k=1}^m \beta_k \left[ q_{ik}^* + \frac{1}{\beta \beta_k} B_i \right]^2 + \frac{1}{\beta} \left[ \frac{1}{r_0 - r - 1} \sum_{h \in I_0} B_h \right] A_i = \\ &= \sum_{k=1}^m \beta_k [q_{ik}^*]^2 + \frac{1}{\beta} \left[ \left( \frac{1}{r_0 - r - 1} \sum_{h \in I_0} B_h + B_i \right) A_i - B_i^2 \right] \end{aligned}$$

Observe that at equilibrium the profit achieved by all firms with  $B_i \geq 0$  is greater than the profit that they would obtain in the traditional Cournot game if all firms played the Cournot quantities. In particular, if  $B_i = 0$  then firm  $i$  delivers the Cournot quantities in the constrained game, but it achieves a profit greater than the profit obtained in the traditional Cournot game.

On the contrary, if  $B_i < 0$  then firm  $i$  achieves a profit smaller than the profit obtained in the traditional Cournot game only if it is verified that:

$$\frac{1}{\beta} \left[ \left( \frac{1}{r_0 - r - 1} \sum_{h \in I_0} B_h + B_i \right) A_i - B_i^2 \right] < 0.$$

The previous inequality holds either if

$$\frac{1}{r_0 - r - 1} \sum_{h \in I_0} B_h < -B_i,$$

or if  $A_i$  is small enough. Note that, the smaller capacity of production  $A_i$ , the greater value  $-B_i$  is obtained. Otherwise, in spite of producing quantities smaller than the Cournot quantities, firm  $i$  gets a profit equal to, or greater than, the one gained in the traditional Cournot game.

## 5. Concluding remarks

We have determined the exact Nash equilibrium for the Cournot game when a maximum quantity of product can be delivered by each competing firm. Analytical expressions for quantities have been obtained, which are the exact equilibrium quantities under some conditions. These conditions hold true when both Cournot quantities are large enough for those firms that cannot produce the total Cournot quantity and the maximum quantity that can be produced is large enough for those firms that can produce the total Cournot quantity.

In the constrained game, the profit gained by one firm, which can produce the total Cournot quantity, is always greater than the profit the firm would obtain in the traditional Cournot game. It is remarkable that, if for one firm the Cournot quantities are the equilibrium quantities in both games, then the firm gains a greater profit in the constrained game. That is what happens when the maximum quantity equals the total Cournot quantity. In other cases, the profit at equilibrium of one firm may be greater, or not, than that of the traditional Cournot game.

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